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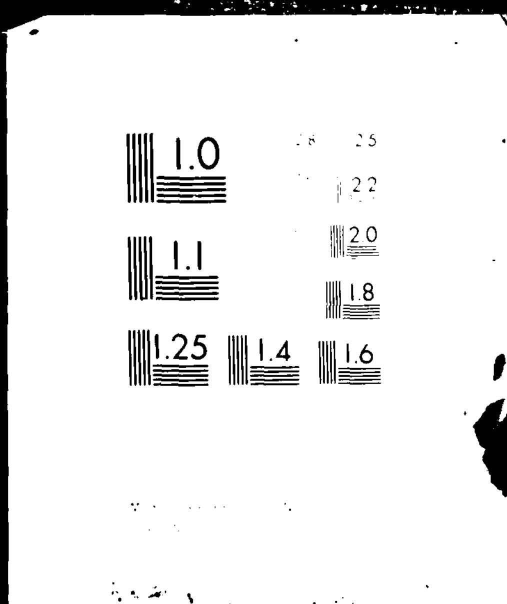

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 82-0222	2. GOVT ACCESSION NO. AD-A112494
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) LASER INDUCED FORCED MOTION AND STRESS WAVES IN PLATES AND SHELLS	5. TYPE OF REPORT & PERIOD COVERED FINAL 1 OCT 75 - 30 SEP 80
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) HERBERT REISMANN	8. CONTRACT OR GRANT NUMBER(s) AFOSR-76-2943
9. PERFORMING ORGANIZATION NAME AND ADDRESS DEPARTMENT OF MECHANICAL & AEROSPACE ENGINEERING STATE UNIVERSITY OF NEW YORK AT BUFFALO BUFFALO, NY 14260	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2307/B1
11. CONTROLLING OFFICE NAME AND ADDRESS AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA BOLLING AIR FORCE BASE, DC 20332	12. REPORT DATE AUGUST 81
	13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DTIC ELECTE

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Attachment to report is reprints from SM Archives, Vol 2, Issue 2, May 1977, pp 129 - 185. and SM Archives, Vol 6, Issue 2, April 1981, pp 213-277

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

LASER IRRADIATION
THERMOELASTICITY
ELASTIC PLATE THEORIES
DYNAMIC RESPONSE
FLEXURAL DEFLECTION

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A combined analytical and experimental program was conducted to study the forced, thermoelastic motion of plates due to laser irradiation. The first part of the theoretical study deals with modeling and analysis of a simply supported rectangular plate irradiated by a laser beam at an arbitrary point. The initial phase of the response is modeled as a localized effect in the vicinity of the irradiated area. It is shown that in this area, a dilation wave is set up which propagates in the direction of the plate thickness. This causes periodic in-plane stresses (tension and compression) which are however

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ll (by at least 2 orders of magnitude) compared to the gross flexural motions associated with the latter phase of the plate response. The flexural response was predicted using three different theories: (a) three-dimensional elasticity theory, (b) classical plate theory, and (c) improved plate theory (incorporating shear deformation and rotatory inertia effects). All three gave the same time history response with less than 0.1% differences among them in the amplitude of vibration. The second part of the analytical study involves the boundary value problem of a clamped circular plate subject to normal laser irradiation at the center. An exact solution of the dynamic response was obtained via a series representation. The results provided the theoretical base for the experimental validation effort. Experiments were conducted on circular steel and aluminum thin plates, clamped around the boundary, and irradiated at the center with a laser beam of approximately 3 joules power output and 40×10^{-6} seconds pulse width. Plate center deflections were measured as a function of time and compared with the theoretical predictions. The agreement was excellent in terms of frequency and phase relationships. Vibration amplitudes however, differed between theory and experiment by amounts not exceeding 20% of the maximum observed deflection.

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AFOSR-TR- 82 -0222

AFOSR Final Scientific Report

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Grant -AFOSR-76-2943

"Laser Induced Forced Motion and Stress Waves in Plates & Shells"

RESEARCH SUMMARY

for the Period
October 1, 1975 - September 30, 1980

by

Herbert Reismann
Principal Investigator

Department of Mechanical and Aerospace Engineering

STATE UNIVERSITY OF NEW YORK AT BUFFALO
Buffalo, New York 14260

August 1981

Approved for public release; distribution unlimited.

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AFOSR Final Scientific Report

Summary of Research Accomplishments
For the Period
October 1, 1975 - September 30, 1980

by

Herbert Reismann
Principal Investigator

AFOSR Final Report
August 1981

Grant AF-AFOSR-76-2943

Research Sponsored by

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Office of Aerospace Research
United States Air Force
Washington D.C. 20332

ABSTRACT

The following is a summary of research activities for the period October 1, 1975 to September 30, 1980, related to grant AF-AFOSR-76-2943. The primary purpose of the research was the study of forced, thermoelastic motion of plates due to Laser irradiation. This summary describes the theoretical and experimental phases of the research project.

TABLE OF CONTENTS

1.0 Introduction

One of the primary missions of the U.S. Air Force is to provide leadership in weapon system development, and to be cognizant of any and all advances in pure and applied scientific knowledge which has a bearing on changes in the state of the art. Current developments centering on Laser technology have advanced to a stage where Lasers must be considered as potential weapons. A high power Laser has the ability to concentrate a short duration, high energy flux in a very narrow beam. Thus it has the ability to deposit focused radiant energy upon opaque solids or structural components. Unlike other, conventional weapons, it is not dependent upon a ballistic path, but it is inherently a line of sight weapon. Because radiation travels with the speed of light (approximately 3×10^{10} cm/sec) such a weapon can inflict almost instantaneous damage to any target upon which it is trained. These characteristics obviate the need for a ballistic computer as well as complex lead aiming devices, and the time between target acquisition and radiation impact is negligible.

The present study was concerned with the interaction of a laser beam with the skin of an aircraft, a re-entry vehicle, or a satellite, i.e., a thin plate or shell. Laser interaction with plates or shells, in general is a complex phenomenon. For purposes of experimentation and associated analysis, it is possible to define three types of interaction resulting in (more or less) separated effects:

(a) Sudden deposition of thermal energy, without a change in phase. This causes sudden thermal stresses in the irradiated plate. Because of the rapidity of the energy deposition process, there will be thermally generated stress waves.

(b) Surface vaporization of a very thin layer of material, plasma production, plasma heating, shockwave formation in plasmas, etc. This mode of interaction results in suddenly applied surface pressures of considerable magnitude to the solid plate, inducing time-dependent stress waves and deformations.

(c) Complete local vaporization of the material, and the resulting creation of openings (holes or large surface cavities).

It is to be noted that conditions (a), (b), and (c) usually coexist, but they can be (nearly) separated by proper choice of laser and target parameters.

Laser technology and laser weapons development is an on-going activity in the U.S.A. and in other countries. By virtue of its mission, the U.S. Air Force must keep abreast of such developments, and must be prepared to evaluate accurately the potential and destructive capabilities of such weapons when and if they become operational. In addition, detailed knowledge of the capabilities of such a weapon system will undoubtedly suggest ways and means to either avoid, circumvent, or reduce its destructive effects upon potential targets.

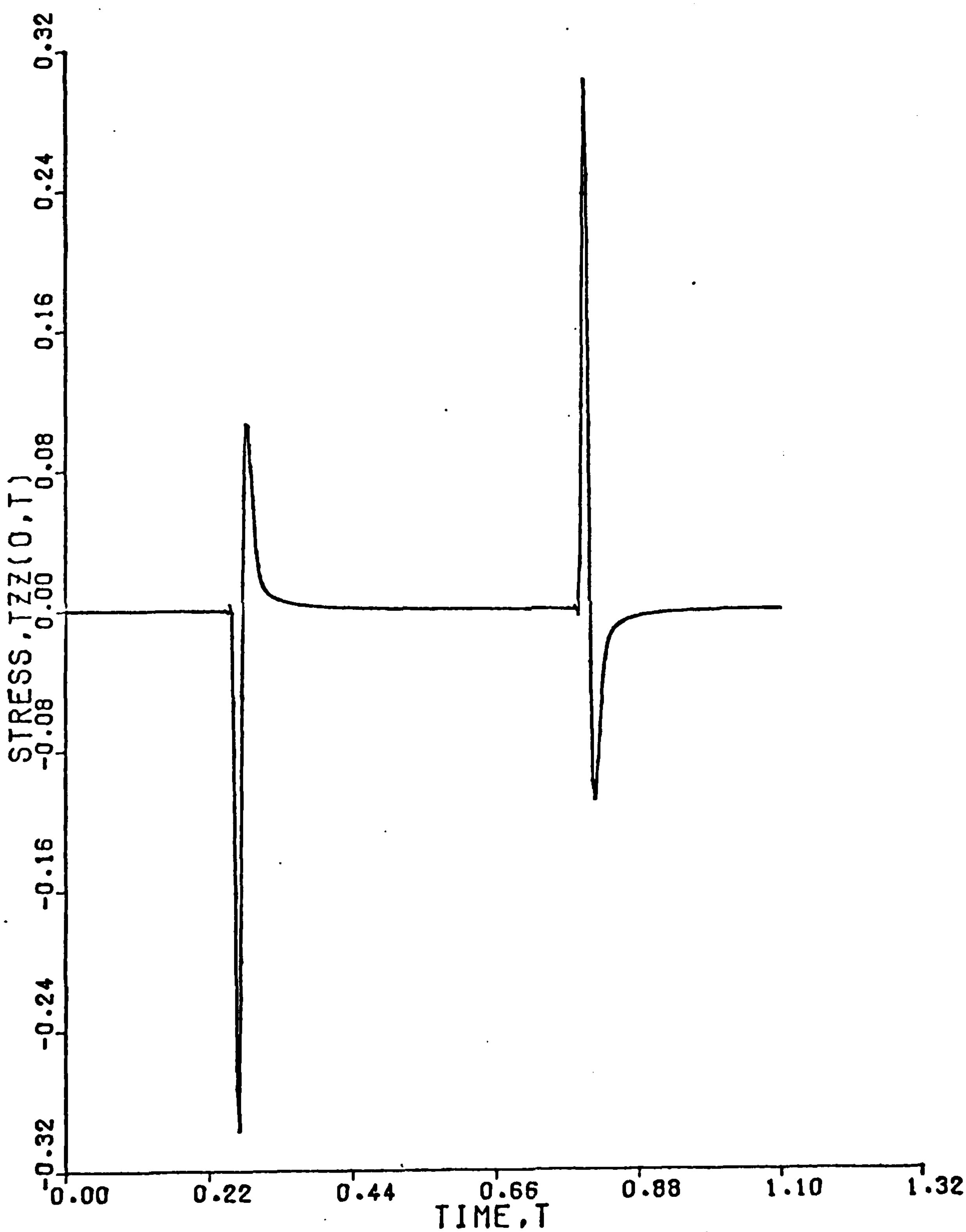

FIG. 1 : Normal Stress at the Middle Surface vs. Time.

2.0 Summary of Research Accomplishments

2.1 Theoretical

Detailed and comprehensive theoretical studies were performed to model the motion of a rectangular plate when subjected to laser irradiation (see references 2 and 4). In these studies, we utilized the Bechtel model for Laser heating.

The first part of our study deals with the thickness-stretch motion of a transversely constrained, irradiated slab. This part of the study models the initial response of the plate. The initial motion of the plate is predominantly thickness-stretch in the vicinity of the irradiated area. In this area, stresses and displacements are primarily in the direction of plate thickness. However, these stresses and displacements are relatively small compared to the stresses and displacements resulting from the gross (predominantly flexural) motion of the plate. It is shown that in the vicinity of the Laser beam, a dilatational wave is set up which moves in the direction of the plate thickness. This wave causes periodic tension and compression stresses on the median surface of the plate (see Fig. 1).

The second part of this study is concerned with the gross-motion of the rectangular plate which is assumed to be simply supported along its boundaries. The plate surface is irradiated by a Laser beam at an arbitrary point. Three different theories are used to model the time-dependent, thermoelastic motion of the plate:

(a) Three-dimensional elasticity theory (Ref. 2)

(b) Classical Plate Theory (Ref. 4)

(c) Improved Plate Theory (including the effects of shear deformation and rotatory inertia). (Ref. 4)

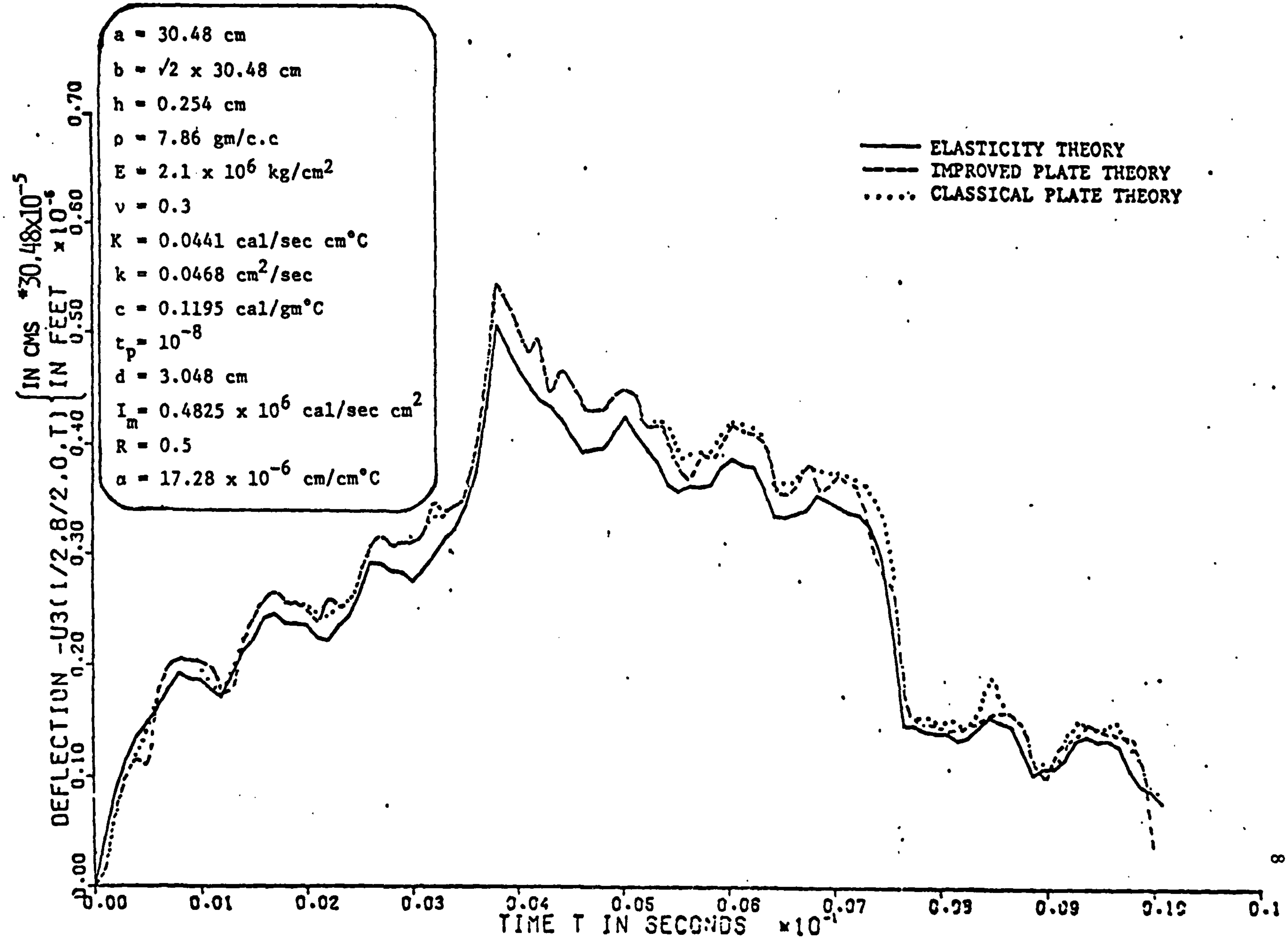

FIG. 2. Comparison of I.P.T. and C.P.T. with Elasticity Solution

In each case, an exact solution (in series form) was found for the boundary value problem. A comparison of the three solutions reveals essentially equivalent results for the gross motion (deflection, bending moment, etc.). In this connection, see Fig. 2. This implies that future calculations for practical plate (or shell) structures subject to Laser irradiation can be carried out within the framework of relatively simple mathematical models, resulting in a considerable reduction of the computational effort. Fig. 2 shows the time history of the center deflection of an irradiated plate. The three curves correspond to the three different theories employed for the computation.

In references 1 and 3 we consider the boundary value problem of the clamped circular plate subject to normal Laser irradiation at the center of the plate. An exact solution for the dynamic response of the plate was found in series form. This model includes the effects of flexure, shear, transverse, rotatory and radial inertia forces, etc. The results of this study served as the theoretical base for experiments described below.

2.2 Experimental

Thin aluminum and steel plates in the shape of a circle (radius $\approx$ 11.5 cm) were subjected to laser irradiation in the laboratory. The plate was clamped at the boundary with the aid of a heavy, machined ring fixture. The laser beam was directed to intercept the plate at the center, normal to the plate surface. The Laser used was a Holobeam model 630-QNd glass system. This Laser produces an output power (in the Q switched mode) of approximately 3 joules, with a pulse width of approximately 40×10^{-9} sec.

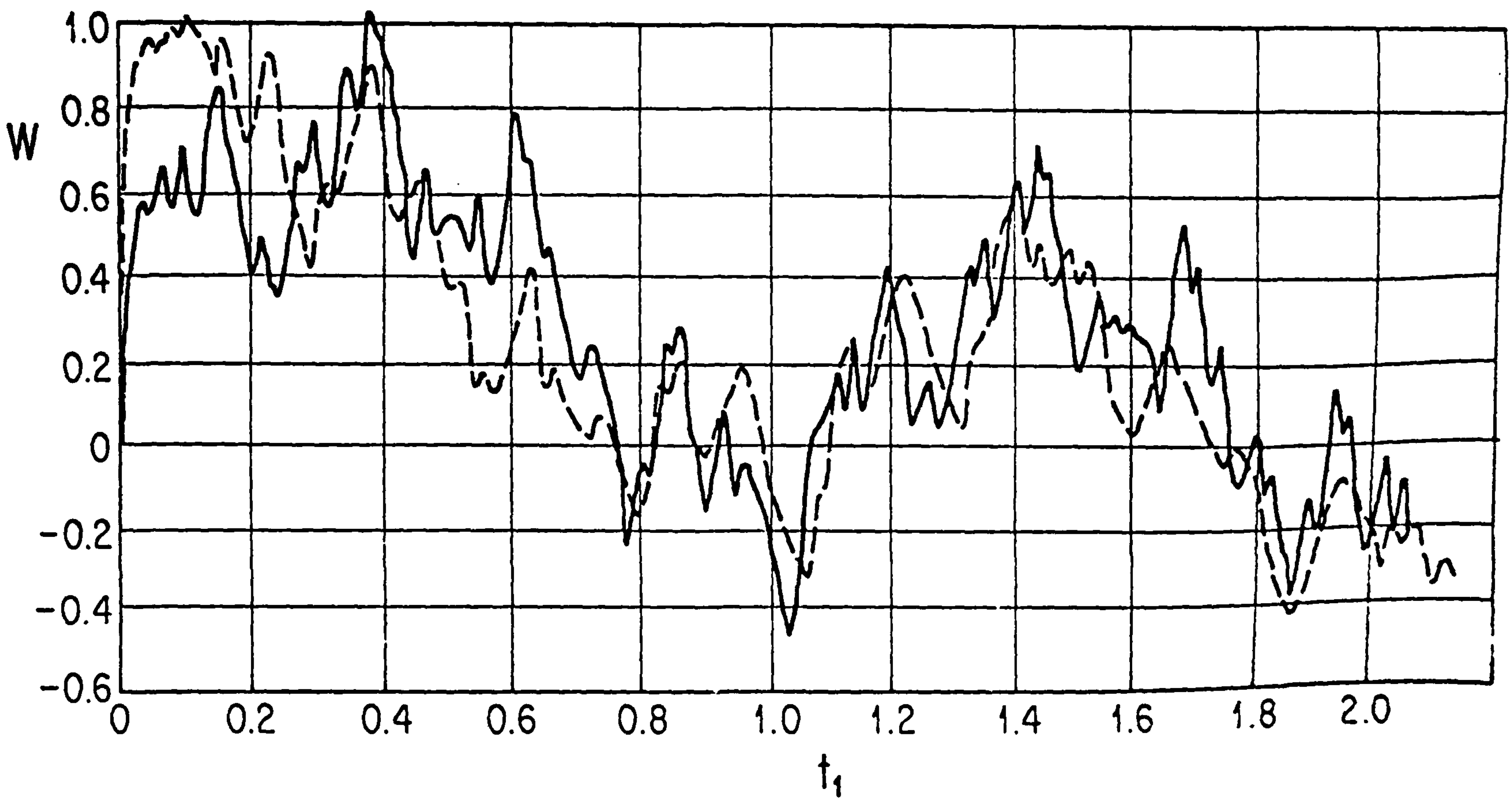

Figure 3 : Comparison of Theory and Experimental Data

Horizontal Scale – Units of $T=1/f_0$, f_0 = fundamental frequency (120 Hz)
Vertical Scale – Experimental Data expanded by x 2.
Theoretical Scale Arbitrary

Theoretical Data ———
Experimental Data - - - -

The output beam is about 1 cm in diameter with a beam divergence of 2 m radians, at an output wave length of 1.06 μ. The beam deposits radiant energy onto the metal plate, resulting in heating. Time dependent thermo-elastic stresses are relieved by the motion of the plate.

The motion of the plate is sensed by a Bruel and Kjaer model MM004 capacitive transducer and associated circuitry. The amplified signal is displayed on a Tektronix 545B oscilloscope and photographed. The detector was mounted at a separation of 1 mm from the rear surface (away from the laser), at the center of the plate.

The results of experiments (plate center displacement vs. time) are shown in Fig. 3. The theoretical curve is also shown in Fig. 3, and it can be concluded that the mathematical model predicts the motion with reasonable accuracy.

In addition to the above experimental work, exploratory work was performed to move the plate by impulsive means. Lasers were used to vaporize a very thin layer of the plate metal surface material. This produces a plasma which is heated (by the Laser), and subsequently produces a shock wave. The shock wave impinges upon the plate surface and causes the plate to move impulsively. A mathematical model for this type of pressure loading has been considered (see reference 7).

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Offprint from

SM ARCHIVES

SM Archives, Vol. 6, Issue 2, April 1981
Sijthoff & Noordhoff International Publishers
Alphen aan den Rijn
Printed in The Netherlands

WAVE MOTION IN NON-HOMOGENEOUS BEAMS AND PLATES — A COMPARISON OF TWO THEORIES

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(Received January 1981)

PART I

1. INTRODUCTION

It is well known that Euler-Bernoulli beam theory and its two-dimensional counterpart, classical plate theory, neglect the effects of rotatory inertia and transverse shear deformations. As a consequence of this omission, results obtained by these classical (approximate) theories are valid only for the case of waves which are long compared to the radius of gyration of the beam cross-section or the plate thickness. On the other hand, classical elasticity theory, while still limited to sufficiently small deformations, imposes no restriction upon the wave length ratio, but solutions for beam and plate dynamics problems within the framework of the exact theory require a monumental computational effort, if such solutions can be obtained at all.

In the case of beams, correction terms have been supplied by Rayleigh [1] and Timoshenko [2,3]. Rayleigh introduced the effect of rotatory inertia and Timoshenko included also the effect of transverse shear deformation. Thus, this one-dimensional theory well known as Timoshenko beam theory occupies a position intermediate between Euler-Bernoulli beam theory and elasticity theory and is expected to give satisfactory results for short waves.

In the case of plates, equations of motion analogous to Timoshenko's beam equations have been given by Mindlin [4] and Uflyand [5], and a corresponding theory of plate equilibrium has been given by Hencky [6] and Reissner [7].

Since these improved theories and elementary theories are characterized by differing dispersion relations, we can expect that certain wave reflection and transmission problems associated with piecewise non-homogeneous plates and beams will result in major differences in the predictions of the two theories.

The present investigation is divided into two parts:

(1) Transmission and reflection of waves in piecewise non-homogeneous beams. In this case we consider wave motion in two bonded, semi-infinite beams composed of different materials, and also the case of a beam of finite length bonded at each end to two semi-infinite beams composed of a different material.

(2) Transmission and reflection of waves in piecewise non-homogeneous plates the construction of which are similar to part (1), except that we consider wave motion in plates instead of beams.

2. BASIC EQUATION OF BEAMS

2.1 *Description of the Motion*

We shall assume the following displacement components for a beam:

$$\begin{aligned} U_x &= z\,\phi(x,t) \\ U_y &= 0 \\ U_z &= w(x,t) \quad . \end{aligned} \tag{1.1}$$

The strain-displacement relationship can be written as follows:

$$
\begin{aligned}
e_{xx} &= z\,\frac{\partial\phi}{\partial x} \\
e_{yy} &= 0 \\
e_{zz} &= 0 \\
e_{xz} &= \frac{1}{2}\left(\phi + \frac{\partial w}{\partial x}\right) \\
e_{yx} &= 0 \\
e_{zy} &= 0 \quad .
\end{aligned}
\tag{1.2}
$$

2.2 *Energy Considerations and Hamilton's Principle*

In the theory of small motions of an elastic isotropic continuum, the kinetic and potential energies of deformation are

$$
\begin{aligned}
T &= \frac{1}{2}\int_0^\ell\!\int_A \rho(\dot{U}_x^2 + \dot{U}_y^2 + \dot{U}_z^2)\,dx\,dA \\
&= \frac{1}{2}\int_0^\ell\!\int_A \rho(z^2\,\dot{\phi}^2 + \dot{W}^2)\,dx\,dA \quad .
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
V &= \frac{1}{2}\int_0^\ell\!\int_A (\tau_{xx}e_{xx} + \tau_{yy}e_{yy} + \tau_{zz}e_{zz} \\
&\qquad + 2\tau_{xz}e_{xz} + 2\tau_{xy}e_{xy} + 2\tau_{yz}e_{yz})\,dx\,dA \\
&= \frac{1}{2}\int_0^\ell\!\int_A \left[\tau_{xx}z\,\frac{\partial\phi}{\partial x} + \tau_{xz}\left(\phi + \frac{\partial w}{\partial x}\right)\right]dx\,dA
\end{aligned}
\tag{1.4}
$$

where a dot indicates partial differentiation with respect to time. Define the moment of inertia, the area of the section, the bending moment and shearing force as

$$
\begin{aligned}
I &= \int_A z^2\,dA \\
A &= \int_A dA
\end{aligned}
\tag{1.5}
$$

$$M = \int_A \tau_{xx} z \, dA$$
$$Q = \int_A \tau_{xz} \, dA \quad . \tag{1.6}$$

Then equations (1.3) and (1.4) reduce to

$$T = \frac{1}{2} \int_0^\ell \rho (I \, \dot{\phi}^2 + A \, \dot{w}^2) dx \tag{1.7}$$

$$V = \frac{1}{2} \int_0^\ell \left[M \frac{\partial \phi}{\partial x} + Q \left(\phi + \frac{\partial w}{\partial x} \right) \right] dx \quad . \tag{1.8}$$

The variation of T and v are

$$\delta T = \int_0^\ell \rho (I \, \dot{\phi} \, \delta \, \dot{\phi} + A \, \dot{w} \, \delta \, \dot{w}) \, dx \tag{1.9}$$

$$\delta V = \int_0^\ell \left[M \frac{\partial}{\partial x} (\delta \phi) + Q \left\{ \delta \phi + \frac{\partial}{\partial x} (\delta w) \right\} \right] dx \quad . \tag{1.10}$$

Denote the work done by surface tractions when the displacements are varied by δW. Then

$$\delta W = \int_0^\ell (m \delta \phi + q \delta w) dx + [M^* \delta \phi + Q^* \delta w]_{x=0}^{x=\ell} \tag{1.11}$$

where

$$m = \int_{-\frac{v}{2}}^{\frac{v}{2}} [\tau_{xz} z]_{-\frac{h}{2}}^{\frac{h}{2}} dy$$

$$q = \int_{-\frac{v}{2}}^{\frac{v}{2}} [\tau_{zz}]_{-\frac{h}{2}}^{\frac{h}{2}} dy$$

$$M^* = \int_A \tau_{xx} z \, dA \qquad \text{at } x=0 \text{ or } \ell$$

$$Q^* = \int_A \tau_{xz} \, dA \qquad \text{at } x=0 \text{ or } \ell$$

Hamilton's principle in an interval of time t_1 to t_2 is

$$\int_{t_2}^{t_1} (\delta T - \delta V + \delta W)dt = 0 \tag{1.12}$$

After substitution of equations (1.9), (1.10) and (1.11) into (1.12) and upon application of integration by parts, we are led to the following equation

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{o}^{\ell}\left[(-\rho A\ddot{w} + \frac{\partial Q}{\partial x} + q)\delta W + (-\rho I\ddot{\phi} + \frac{\partial M}{\partial x} - Qm)\delta\phi\right]dx\ dt \\ &+ \int_{t_1}^{t_2}\left[(M^*-M)\delta\phi + (Q^*-Q)\delta W\right]_{x=o}^{x=\ell} dt = 0 \ .\end{aligned} \tag{1.13}$$

2.3 Equations of the Motion

From equation (1.13), the equations of motion can be written as follows:

$$\begin{aligned}&-\rho A\ddot{w} + \frac{\partial Q}{\partial w} + q = 0 \\ &-\rho I\ddot{\phi} + \frac{\partial M}{\partial x} - Q+m = 0 \ .\end{aligned} \tag{1.14}$$

The appropriate boundary conditions are:

1) At each end of the beam, $x=0$ and $x=\ell$, one member of each of the pairs, (M,ϕ) and (Q,w) must be specified.
2) At the surface of the beam, $z = \pm \frac{h}{2}$, one member of each of the pairs (m,ϕ) and (q,w) must be specified.

To insure a unique solution the following initial conditions are required:

$w(x,0)$, $\phi(x,0)$, $\dot{w}(x,0)$ and $\dot{\phi}(x,0)$ are specified.

The general Hook's law reduces for the present isotropic case to

$$E\, e_{xx} = \tau_{xx} - \nu(\tau_{yy} + \tau_{zz})$$

$$2G\, e_{xz} = \tau_{xz} \quad . \tag{1.15}$$

Upon substitution of equations (1.2) into (1.15), we obtain

$$\tau_{xx} = E\, z\, \frac{\partial \phi}{\partial x} + \nu(\tau_{yy} + \tau_{zz})$$

$$\tau_{xz} = G\, \left(\phi + \frac{\partial w}{\partial x}\right) \quad . \tag{1.16}$$

Further substitution of equations (1.16) into (1.6), and using Timoshenko's correction factor κ, the beam stress displacement relations become

$$M = E\, I\, \frac{\partial \phi}{\partial x}$$

$$Q = \kappa^2\, A\, G\, \left(\phi + \frac{\partial w}{\partial x}\right) \tag{1.17}$$

where

$$\nu \int_A z\, (\tau_{yy} + \tau_{zz})\, dA$$

is ignored.

The displacement equations of motion are now obtained by substitution of (1.17) into (1.14), with the result

$$\kappa^2\, A\, G\, \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \phi\right) + q = \rho A \ddot{w}$$

$$EI\, \frac{\partial^2 \phi}{\partial x^2} - \kappa^2\, A\, G\, \left(\frac{\partial w}{\partial x} + \phi\right) + m = \rho I \ddot{\phi} \quad . \tag{1.18}$$

Equations (1.18) are the well known Timoshenko Beam equations.

2.4 *Energy Flux*

With reference to equations (1.17), the sum of kinetic and potential energies for a beam of length $x_1 - x_2$ is

$$T + V = \int_{x_1}^{x_2} \left[\rho(I\dot{\phi}^2 + A\dot{w}^2) + EI\left(\frac{\partial\phi}{\partial x}\right)^2 + \kappa^2 AG\left(\phi + \frac{\partial w}{\partial x}\right)^2 \right] dx. \tag{1.19}$$

After differentiation of (1.19) with respect to time and upon application of integration by parts, we obtain

$$\frac{\partial}{\partial t}(T+V) = \int_{x_1}^{x_2} \left[\dot{w}\left(\rho A\ddot{w} - \frac{\partial Q}{\partial x}\right) + \dot{\phi}\left(\rho I\ddot{\phi} - \frac{\partial M}{\partial x} + Q\right)\right] dx \quad . \\ + \left[M\dot{\phi} + Q\dot{w}\right]_{x_2}^{x_1} \tag{1.20}$$

When there are no applied loads, $q = m = 0$, and the integrand of the right hand side is identically zero. Thus the equation reduces to

$$\frac{\partial}{\partial t}(T+V) = \left[M\dot{\phi} + Q\dot{w}\right]_{x_1}^{x_2} \quad . \tag{1.21}$$

We identify the energy flux as

$$J(x,t) = -\left[M\dot{\phi} + Q\dot{w}\right] \quad . \tag{1.22}$$

Upon substitution of equations (1.17) into (1.22) we obtain

$$J(x,t) = -\left[EI\,\frac{\partial\phi}{\partial x}\,\dot{\phi} + \kappa^2 AG\left(\frac{\partial w}{\partial x} + \phi\right)\dot{w}\right] \quad . \tag{1.23}$$

2.5 *Reduction from Timoshenko Beam Theory to Euler-Bernoulli Beam Theory*

If we let $\phi = -\frac{\partial w}{\partial x}$ in the equation (1.13), we are led to the following equation

$$\int_{t_1}^{t_2}\int_{0}^{\ell}\left\{\frac{\partial^2 M}{\partial x^2} + q - \rho A\ddot{w} + \rho I\,\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial m}{\partial x}\right\}\delta w\,dx\,dt$$
$$+ \int_{t_1}^{t_2}\left[-(M^*-M)\,\frac{\partial w}{\partial x} + (Q^*-Q)\delta w\right]_{x=0}^{x=\ell} dt \qquad (1.24)$$
$$- \int_{t_1}^{t_2}\left[\left(\rho I\,\frac{\partial \ddot{w}}{\partial x} + \frac{\partial M}{\partial x} - Q + m\right)\delta w\right]_{x=0}^{x=\ell} dt = 0\ .$$

Thus the equation of motion can be written as follows

$$\frac{\partial^2 M}{\partial x^2} + q - \rho A\ddot{w} + \rho I\,\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial m}{\partial x} = 0\ .$$

If we delete the rotatory inertia $\rho I\,\frac{\partial^2 \ddot{w}}{\partial x^2}$ and also $\frac{\partial m}{\partial x}$ we obtain

$$\frac{\partial^2 M}{\partial x^2} + q = \rho A\ddot{w}\ .$$

Since the third term of the equation (1.24) vanishes because of the second equation of (1.14), the appropriate boundary conditions are as follows.

At each end of the beam $x=0$ and $x=\ell$, one of each of the products $(M, \frac{\partial w}{\partial x})$ and (Q,w) must be specified.

And also from the second equation of (1.14), we obtain

$$Q = \frac{\partial M}{\partial x} + \rho I\,\frac{\partial \ddot{w}}{\partial x} + m\ .$$

If we delete $\rho I\,\frac{\partial \ddot{w}}{\partial x}$ and m, we obtain

$$Q = \frac{\partial M}{\partial x}\ .$$

The energy flux equation in this case reduces to

$$J(x,t) = -\left[-M\,\frac{\partial \dot{w}}{\partial x} + Q\dot{w}\right]\ . \qquad (1.25)$$

The displacement equations of motion and moment and shear force are obtained as follows:

$$
\begin{aligned}
&-EI \frac{\partial^4 w}{\partial x^4} + q = \rho A \ddot{w} \\
&M = - EI \frac{\partial^2 w}{\partial x^2} \qquad\qquad (1.26) \\
&Q = - \frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2} \right) \quad .
\end{aligned}
$$

3. WAVE PROPAGATION IN PIECEWISE NON-HOMOGENEOUS BEAMS

3.1 *General Solution*

Equations (1.18) with $q = m = 0$ reduce to

$$
\begin{aligned}
&\kappa^2 AG \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \phi \right) = \rho A \ddot{w} \\
&EI \frac{\partial^2 \phi}{\partial x^2} - \kappa^2 AG \left(\frac{\partial w}{\partial x} + \phi \right) = \rho I \ddot{\phi} \quad .
\end{aligned}
\qquad (1.27)
$$

Assume

$$
\begin{aligned}
&w(x,t) = A\, e^{i\Omega(t - \frac{x}{C})} \\
&\phi(x,t) = D\, e^{i\Omega(t - \frac{x}{C})}
\end{aligned}
\qquad (1.28)
$$

where Ω is wave frequency and C is phase velocity. Upon substitution of equations (1.28) into (1.27) we obtain a pair of homogeneous, linear algebraic equations in A and D whose determinant, set equal to zero, yields the velocity equation

$$
(1-a)\, C^4 - (Ce^2 + Cs^2)\, C^2 + C_s^{\,2}\, C_e^{\,2} = 0 \qquad (1.29)
$$

where

$$
C_e^2 = \frac{E}{\rho} , \quad C_s^2 = \frac{\kappa^2 G}{\rho} , \quad a = \frac{A\kappa^2 G}{I\Omega^2 \rho} = \frac{1}{\Omega^2} \cdot \frac{A}{I}\, C_s^2 \quad .
$$

From equation (1.29), we obtain the mode velocities as follows:

$$C_1^2 = \frac{C_e^2 + C_s^2 - [(C_e^2 + C_s^2)^2 - 4(1-a)C_e^2 C_s^2]^{1/2}}{2(1-a)}$$

$$C_2^2 = \frac{C_e^2 + C_s^2 + [(C_e^2 + C_s^2)^2 - 4(1-a)C_e^2 C_s^2]^{1/2}}{2(1-a)} \quad . \tag{1.30}$$

The most general solution of (1.27) is therefore

$$w = \sum_{k=1}^{2} \left\{ A_k e^{i\Omega(t - \frac{x}{C_k})} + B_k e^{i\Omega(t + \frac{x}{C_k})} \right\}$$

$$\psi = \sum_{k=1}^{2} \left\{ D_k e^{i\Omega(t - \frac{x}{C_k})} + R_k e^{i\Omega(t + \frac{x}{C_k})} \right\} \tag{1.31}$$

where it is understood that the frequencies must be the same for all waves and that C_2 will become imaginary as $a > 1$. A plot of phase velocity vs. frequency is shown in Figure 1.1. Note that of the 8 coefficients appearing in equation (1.31), only 4 are independent. After substitution of equations (1.31) into (1.27) we find that

$$A_k = P_k D_k i \quad , \qquad k = 1,2$$

$$B_k = - P_k R_k i \quad , \qquad k = 1,2 \tag{1.32}$$

where

$$P_k = \frac{C_k C_s^2}{\Omega(C_k^2 - C_s^2)} \,, \qquad k = 1,2 \quad .$$

In Euler-Bernoulli Beam Theory, equation (1.26) with $q=0$ reduces to

$$EI \frac{\partial^4 w}{\partial x^4} = - \rho A \ddot{w} \quad . \tag{1.33}$$

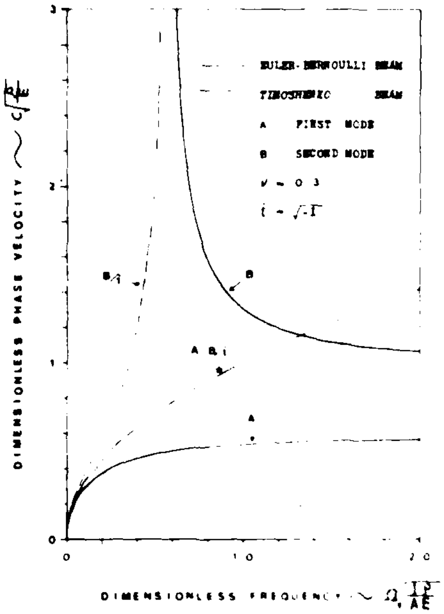

Figure 1.1 – [illegible]

The most general solution of (1.33) is therefore

$$w = \sum_{k=1}^{2} \left\{ A_k e^{i\Omega\left(t - \frac{x}{C_k}\right)} + B_k e^{i\Omega\left(t + \frac{x}{C_k}\right)} \right\} \tag{1.34}$$

where

$$\begin{aligned} C_1^2 &= \left(\frac{EI}{\rho A}\right)^{1/2} \Omega \\ C_2^2 &= -\left(\frac{EI}{\rho A}\right)^{1/2} \Omega = -C_1^2 \ . \end{aligned} \tag{1.35}$$

Since C_2 is always imaginary, A_2 must be equal to zero to insure a bounded solution for $x \to -\infty$. A plot of phase velocity vs. frequency is shown in Figure 1.1.

3.2 *Wave Motion in Two Bonded Semi-Infinite Beams Composed of Different Materials*

Two semi-infinite beams of different materials are bonded at $x=0$, as shown in Figure 1.2. For a disturbance coming from the negative x direction, let w_1^+ and ϕ_1^+ be the incoming waves, w_1^- and ϕ_1^- be the reflected waves, and let w_2^+ and ϕ_2^+ be the transmitted waves. We have

$$\left.\begin{aligned}
w_1^+(x,t) &= \sum_{k=1}^{2} A_{k1}\, e^{i\Omega(t - \frac{x}{C_{k1}})} \\
\phi_1^+(x,t) &= \sum_{k=1}^{2} D_{k1}\, e^{i\Omega(t - \frac{x}{C_{k1}})} \\
w_1^-(x,t) &= \sum_{k=1}^{2} B_{k1}\, e^{i\Omega(t + \frac{x}{C_{k1}})} \\
\phi_1^-(x,t) &= \sum_{k=1}^{2} R_{k1}\, e^{i\Omega(t + \frac{x}{C_{k1}})}
\end{aligned}\right\} , \quad -\infty < x < 0$$

$$\left.\begin{aligned}
w_2^+(x,t) &= \sum_{k=1}^{2} A_{k2}\, e^{i\Omega(t - \frac{x}{C_{k2}})} \\
\phi_2^+(x,t) &= \sum_{k=1}^{2} D_{k2}\, e^{i\Omega(t - \frac{x}{C_{k2}})}
\end{aligned}\right\} , \quad 0 < x < \infty \qquad (1.36)$$

where

$$A_{kj} = p_{kj} D_{kj}\, i \quad , \qquad B_{kj} = -\, p_{kj} R_{kj}\, i \quad ,$$

$$p_{kj} = \frac{C_{kj}\, C_{sj}^2}{\Omega(C_{kj}^2 - C_{sj}^2)} \quad , \qquad \begin{aligned} k &= 1,2 \\ j &= 1,2 \end{aligned} \quad ,$$

where the subscript $j = 1,2$ refers to the respective domain of the beam (see Figure 1.2), and the subscript $k = 1,2$ refers to the mode of motion.

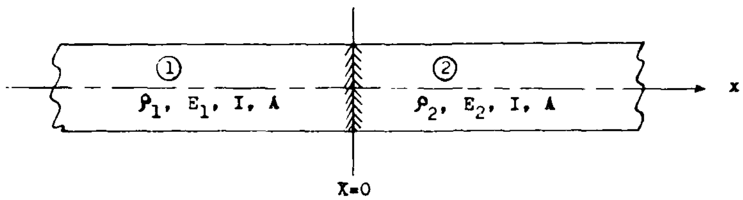

Figure 1.2 - Two Bonded, Semi-Infinite Beams

At the junction x=0, the following four boundary conditions must be satisfied

$$
\begin{aligned}
w_1(0,t) &= w_2(0,t)\\
\phi_1(0,t) &= \phi_2(0,t)\\
Q_1(0,t) &= Q_2(0,t)\\
M_1(0,t) &= M_2(0,t)
\end{aligned}
\tag{1.37}
$$

where

$$Q_j = \kappa^2 A_j G_j \left(\frac{\partial w_j}{\partial x} + \phi_j\right) \quad , \qquad j = 1,2$$

$$M_j = E_j \; I_j \; \frac{\partial \phi_j}{\partial x} \quad , \qquad j = 1,2$$

and where

$$
\begin{aligned}
w_1 &= w_1^+ + w_1^-\\
\phi_1 &= \phi_1^+ + \phi_1^-\\
w_2 &= w_2^+\\
\phi_2 &= \phi_2^+ \quad .
\end{aligned}
$$

Upon substitution of the equations (1.36) into (1.37), we obtain a set of simultaneous, linear, algebraic equations (in matrix form):

$$
\begin{bmatrix}
1 & 1 & -1 & -1 \\
1+b_{11} & 1+b_{21} & -\gamma(1+b_{21}) & -\gamma(1+b_{22}) \\
C_{11}b_{11} & C_{21}b_{21} & C_{12}b_{12} & C_{22}b_{22} \\
\frac{1}{C_{11}} & \frac{1}{C_{21}} & \frac{1}{C_{12}} & \frac{1}{C_{22}}
\end{bmatrix}
\begin{bmatrix} R_{11} \\ R_{21} \\ D_{12} \\ D_{22} \end{bmatrix}
=
\begin{bmatrix}
-D_{11} & -D_{21} \\
-(1+b_{11})D_{11} & -(1+b_{21})D_{21} \\
C_{11}b_{11}D_{11} & +\, C_{21}b_{21}D_{21} \\
\frac{1}{C_{11}} \cdot D_{11} & +\, \frac{1}{C_{21}} \cdot D_{21}
\end{bmatrix}
\tag{1.38}
$$

where

$$
\gamma = \frac{E_2}{E_1} = \frac{G_2}{G_1} , \qquad b_{kj} = \frac{C_{sj}^2}{C_{kj}^2 - C_{sj}^2} = \frac{[illegible]}{C_{kj}} \; p_{kj} .
$$

Once D_{11}, D_{21}, and Ω of the incoming waves and the material properties are known, the quantities R_{11}, R_{21}, D_{12} and D_{22} are obtained by the application of Cramer's Rule. When $a > 1$, D_{21} must be equal to zero to insure a bounded solution for $x \to -\infty$.

The energy flux of the incoming wave, reflected wave, and transmitted wave is defined by equation (1.23), except that only real parts of (1.36) are used:

$$
\begin{aligned}
J_1^+ &= -\left[(\mathrm{Re}M_1^+)(\mathrm{Re}\dot{\phi}_1^+) + (\mathrm{Re}Q_1^+)(\mathrm{Re}\dot{W}_1^+)\right] \\
J_1^- &= -\left[(\mathrm{Re}M_1^-)(\mathrm{Re}\dot{\phi}_1^-) + (\mathrm{Re}Q_1^-)(\mathrm{Re}\dot{W}_1^-)\right] \\
J_2^+ &= -\left[(\mathrm{Re}M_2^+)(\mathrm{Re}\dot{\phi}_2^+) + (\mathrm{Re}Q_2^+)(\mathrm{Re}\dot{W}_2^+)\right]
\end{aligned}
\tag{1.39}
$$

The transmission and reflection coefficients are therefore obtained in the following manner:

$$
\begin{aligned}
T &\equiv \frac{\overline{J_2^+(0,t)}}{\overline{J_1^+(0,t)}} \\
R &\equiv \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}}
\end{aligned}
\tag{1.40}
$$

where the bar denotes the time average over a complete period.

From the point of view of conservation of energy, the sum of the transmission and reflection coefficients must be equal to one, i.e. $T + R = 1$. Calculated values of transmission coefficient vs. dimensionless frequency are plotted in Figures 1.3 through 1.6.

In the Euler-Bernoulli Beam Theory, the displacement equations become

$$\left.\begin{aligned} w_1^+ &= A_{11} e^{i\Omega\left(t - \frac{x}{C_{11}}\right)} \\ w_1^- &= B_{11} e^{i\Omega\left(t + \frac{x}{C_{11}}\right)} + B_{21}\, e^{i\Omega\left(t - \frac{x}{C_{21}}\right)} \end{aligned}\right\} -\infty < x < 0 \tag{1.41}$$

$$w_2^+ = A_{12} e^{i\Omega\left(t - \frac{x}{C_{12}}\right)} + A_{22}\, e^{i\Omega\left(t - \frac{x}{C_{22}}\right)} \qquad 0 < x < \infty$$

where

$$C_{1j}^2 = \left(\frac{E_j I_j}{\rho_j A_j}\right)^{1/2} \Omega \quad , \quad C_{2j}^2 = -\left(\frac{E_j I_j}{\rho_j A_j}\right)^{1/2} \Omega = -C_{1j}^2 \quad , \quad j=1,2,$$

At the junction, x=0, the boundary conditions

$$\begin{aligned} w_1(0,t) &= w_2(0,t) \\ \frac{\partial w_1(0,t)}{\partial x} &= \frac{\partial w_2(0,t)}{\partial x} \\ M_1(0,t) &= M_2(0,t) \\ Q_1(0,t) &= Q_2(0,t) \end{aligned} \tag{1.42}$$

where

$$M_j = -E_j I_j \frac{\partial^2 w_j}{\partial^2 x} \quad , \quad j = 1,2$$

$$Q_j = -E_j I_j \frac{\partial^2 w_j}{\partial^2 x} \quad , \quad j = 1,2$$

yield the following set of simultaneous, linear, algebraic equations (in matrix form):

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -i & d & -id \\ 1 & -1 & -\gamma d^2 & \gamma d^2 \\ 1 & i & \gamma d^3 & i\gamma d^3 \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} -A_{11} \\ A_{11} \\ -A_{11} \\ A_{11} \end{bmatrix} \tag{1.43}$$

where

$$d = \left(\frac{\eta}{\gamma}\right)^{1/4}, \quad \gamma = \frac{E_2}{E_1}, \quad \eta = \frac{\rho_2}{\rho_1}$$

Solving the equations (1.43),

$$\begin{aligned} \frac{B_{11}}{A_{11}} &= \frac{2\gamma d(1-d^2)-i(1-\gamma d^2)}{(1+\gamma d^2)^2+2\gamma d(1+d^2)} \\ \frac{A_{12}}{A_{11}} &= \frac{2(1+\gamma d^2)(1+d)}{d\{(1+\gamma d^2)^2+2\gamma d(1+d^2)\}} \quad . \end{aligned} \tag{1.44}$$

Upon substitution of equations (1.44) into (1.41) and application of equation (1.25), we obtain

$$\begin{aligned} \overline{J_1^+(0,t)} &= |A_{11}|^2 \, E_1 I_1 \, \frac{\Omega^4}{C_{11}^3} \\ \overline{J_1^-(0,t)} &= |B_{11}|^2 \, E_1 I_1 \, \frac{\Omega^4}{C_{11}^3} \\ \overline{J_2^+(0,t)} &= |A_{12}|^2 \, E_2 I_2 \, \frac{\Omega^4}{C_{12}^a} \quad . \end{aligned} \tag{1.45}$$

The transmission and reflection coefficients are calculated to be

$$\begin{aligned} T &= \frac{\overline{J_2^+(0,t)}}{\overline{J_1^+(0,t)}} = \frac{4(1+\gamma d^2)^2(1+d)^2\gamma d}{\{(1+\gamma d^2)^2+2\gamma d(1+d^2)\}^2} \\ R &= \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}} = \frac{4\gamma^2 d^2(1-d^2)^2+(1-\gamma d^2)^4}{\{(1+\gamma d^2)^2+2\gamma d(1+d^2)\}^2} \quad . \end{aligned} \tag{1.46}$$

The conservation of energy check is satisfied:

$$T+R = \frac{4(1+\gamma d^2)^2(1+d)^2\gamma d + 4\gamma^2 d^2(1-d^2)^2 + (1-\gamma d^2)^4}{\{(1+\gamma d^2)^2 + 2\gamma d(1+d^2)\}^2} = 1 \qquad (1.47)$$

Numerical values of the transmission coefficient for the Euler-Bernoulli Beam Theory are shown in Figures 1.3 through 1.6 for comparison purposes.

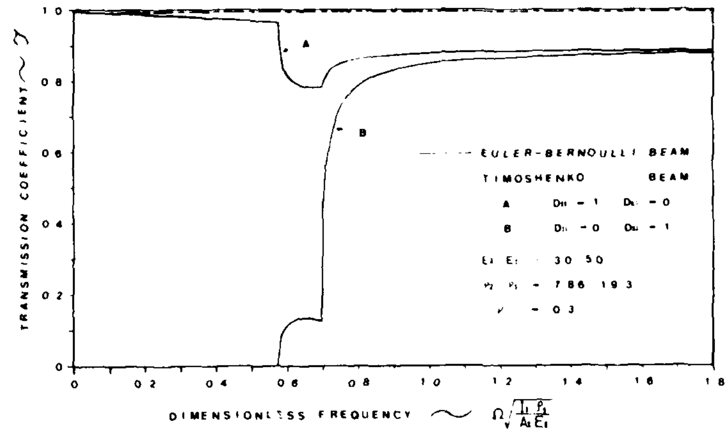

Figure 1.3 - Transmission Coefficient vs. Frequency, Tungsten-Iron Bonded Beams

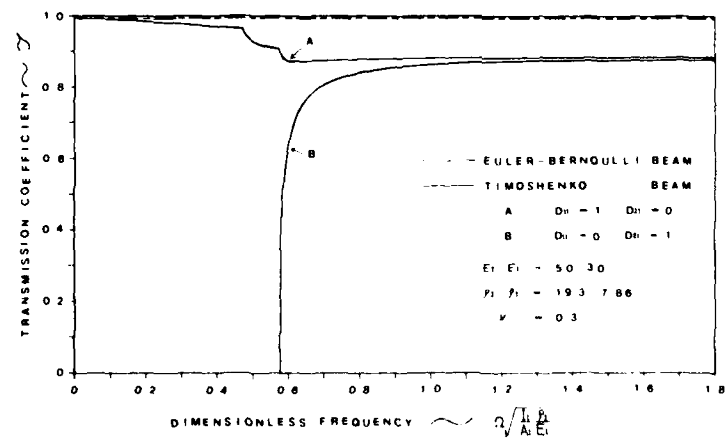

Figure 1.4 - Transmission Coefficient vs. Frequency Iron-Tungsten Bonded Beams

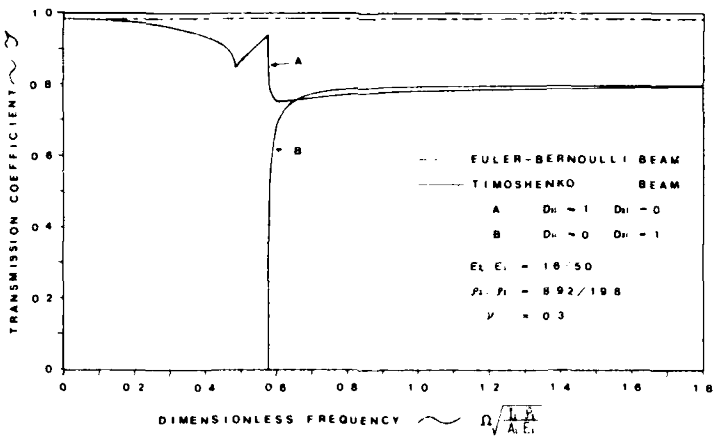

Figure 1.5 - Transmission Coefficient vs. Frequency, Tungsten-Copper Bonded Beams

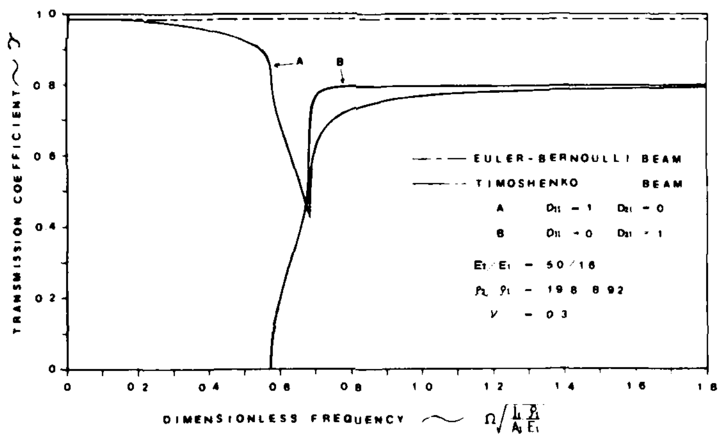

Figure 1.6 - Transmission Coefficient vs. Frequency, Copper-Tungsten Bonded Beams

3.3 *Wave Motion In a Beam of Finite Length Bonded at Each End to Two Semi-Infinite Beams Composed of a Different Material*

A beam of infinite length is bonded at each end to two semi-infinite beams composed of a different material as shown in Figure 1.7.

The displacement equations in this case are:

$$
\left.
\begin{aligned}
w_1^+(x,t) &= \sum_{k=1}^{2} A_{k1}\, e^{i\Omega\left(t - \frac{x}{C_{k1}}\right)} \\
\phi_1^+(x,t) &= \sum_{k=1}^{2} D_{k1}\, e^{i\Omega\left(t - \frac{x}{C_{k1}}\right)} \\
w_1^-(x,t) &= \sum_{k=1}^{2} B_{k1}\, e^{i\Omega\left(t + \frac{x}{C_{k1}}\right)} \\
\phi_1^-(x,t) &= \sum_{k=1}^{2} R_{k1}\, e^{i\Omega\left(t + \frac{x}{C_{k1}}\right)}
\end{aligned}
\right\}, \quad -\infty < x < 0
$$

$$
\left.
\begin{aligned}
w_2^+(x,t) &= \sum_{k=1}^{2} A_{k2}\, e^{i\Omega\left(t - \frac{x}{C_{k2}}\right)} \\
\phi_2^+(x,t) &= \sum_{k=1}^{2} D_{k2}\, e^{i\Omega\left(t - \frac{x}{C_{k2}}\right)} \\
w_2^-(x,t) &= \sum_{k=1}^{2} B_{k2}\, e^{i\Omega\left(t + \frac{x}{C_{k2}}\right)} \\
\phi_2^-(x,t) &= \sum_{k=1}^{2} R_{k2}\, e^{i\Omega\left(t + \frac{x}{C_{k2}}\right)}
\end{aligned}
\right\}, \quad 0 < x < \ell
$$

$$
\left.
\begin{aligned}
w_3^+(x,t) &= \sum_{k=1}^{2} A_{k3}\, e^{i\Omega\left(t - \frac{x-\ell}{C_{k3}}\right)} \\
\phi_3^+(x,t) &= \sum_{k=1}^{2} D_{k3}\, e^{i\Omega\left(t - \frac{x-\ell}{C_{k3}}\right)}
\end{aligned}
\right\}, \quad \ell < x < \infty
$$

(1.48)

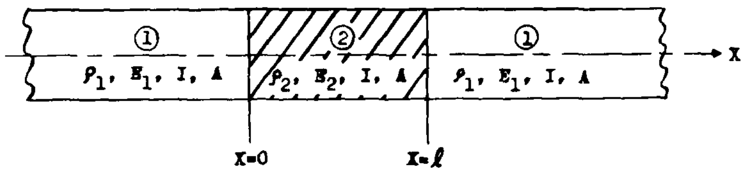

Figure 1.7 - Beam of Finite Length Bonded to Two Semi-Infinite Beams Composed of a Different Material

The relations among the 20 coefficients in equations (1.48) are defined in a manner similar to that of equations (1.36). Note that $C_{13} = C_{11}$, $C_{23} = C_{21}$, $p_{13} = p_{11}$, $p_{23} = p_{21}$, and $b_{13} = b_{11}$, $b_{23} = b_{21}$ for this specific case.

At the junction points, $x = 0$ and $x = \ell$, the following boundary conditions must be satisfied:

$$
\begin{aligned}
w_1(0,t) &= w_2(0,t)\\
\phi_1(0,t) &= \phi_2(0,t)\\
Q_1(0,t) &= Q_2(0,t)\\
M_1(0,t) &= M_2(0,t)\\
w_2(\ell,t) &= w_3(\ell,t)\\
\phi_2(\ell,t) &= \phi_3(\ell,t)\\
Q_2(\ell,t) &= Q_3(\ell,t)\\
M_2(\ell,t) &= M_3(\ell,t)
\end{aligned}
\tag{1.49}
$$

where

$$Q_j = \kappa^2 A_j G_j \left(\frac{\partial w_j}{\partial x} + \phi_j\right) \quad , \qquad j = 1,2,3$$

$$M_j = E_j I_j \frac{\partial \phi_j}{\partial x} \quad , \qquad j = 1,2,3$$

and where

$$
\begin{aligned}
w_1 &= w_1^+ + w_1^- & \phi_2 &= \phi_2^+ + \phi_2^-\\
\phi_1 &= \phi_1^+ + \phi_1^- & w_3 &= w_3^+\\
w_2 &= w_2^+ + w_2^- & \phi_3 &= \phi_3^+
\end{aligned}
$$

Upon substitution of equations (1.48) into (1.49), we obtain a set of simultaneous, linear, algebraic equations as follows:

$$\begin{bmatrix} [1] & [1] & [1] & [0] \\ [1+b_{k1}] & [\gamma(1+b_{k2})] & [\gamma(1+b_{k2})] & [0] \\ [C_{k1}b_{k1}] & [C_{k2}b_{k2}] & [C_{k2}b_{k2}] & [0] \\ [\frac{1}{C_{k1}}] & [\frac{\gamma}{C_{k2}}] & [\frac{\gamma}{C_{k2}}] & [0] \\ [0] & [\varepsilon_k^{-1}] & [\varepsilon_k] & [1] \\ [0] & [\gamma(1+b_{k2})\varepsilon_k^{-1}] & [\gamma(1+b_{k2})\varepsilon_k] & [(1+b_{k3})] \\ [0] & [C_{k2}b_{k2}\varepsilon_k^{-1}] & [C_{k2}b_{k2}\varepsilon_k] & [C_{k3}b_{k3}] \\ [0] & [\frac{\gamma}{C_{k2}}\varepsilon_k^{-1}] & [\frac{\gamma}{C_{k2}}\varepsilon_k] & [\frac{1}{C_{k3}}] \end{bmatrix} \begin{bmatrix} R_{11} \\ R_{21} \\ D_{12} \\ D_{22} \\ R_{12} \\ R_{22} \\ D_{13} \\ D_{23} \end{bmatrix} = \begin{bmatrix} [-D_{k1}] \\ [-(1+b_{k1})D_{k1}] \\ [-b_{k1}D_{k1}C_{k1}] \\ [-\frac{1}{C_{k1}}D_{k1}] \\ [0] \\ [0] \\ [0] \\ [0] \end{bmatrix} \tag{1.50}$$

where, for example

$$\begin{bmatrix} [1+b_{k1}] & & \end{bmatrix} \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix}$$

$$\begin{bmatrix} [-(1+b_{k1})D_{k1}] \end{bmatrix} \equiv \sum_{k=1}^{2} \{-(1+b_{k1})D_{k1}\}$$

and

$$\gamma = \frac{E_2}{E_1} = \frac{E_2}{E_3} = \frac{G_2}{G_1} = \frac{G_2}{G_3} \quad , \quad \varepsilon_k = e^{\frac{i\Omega}{C_{k2}}\ell} \quad , \; k = 1,2$$

$$b_{kj} = \frac{\Omega}{C_{kj}} \quad p_{kj} = \frac{C_{sj}^2}{C_{kj}^2 - C_{sj}^2} \; , \quad k = 1,2 \qquad j = 1,2,3$$

The quantities R_{11}, R_{21}, D_{12}, D_{22}, R_{12}, R_{22}, D_{13}, and D_{23} are obtained by the application of Cramer's Rule.

The transmission and reflection coefficients in this case are defined as:

$$T \equiv \frac{\overline{J_3^+(\ell,t)}}{\overline{J_1^+(0,t)}}$$

$$R \equiv \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}} \tag{1.51}$$

where $J_3^+(x,t)$ is defined in a manner similar to equations (1.39). Two numerical results are shown in Figures 1.8 and 1.9.

In Euler-Bernoulli Beam Theory, the displacement equations are, in this case

$$\left.\begin{aligned} w_1^+ &= A_{11}e^{i\Omega(t - \frac{x}{C_{11}})} \\ w_1^- &= B_{11}e^{i\Omega(t + \frac{x}{C_{11}})} + B_{21}e^{i\Omega(t + \frac{x}{C_{21}})} \end{aligned}\right\}, \qquad -\infty < x < 0$$

$$\left.\begin{aligned} w_2^+ &= A_{12}e^{i\Omega(t - \frac{x}{C_{12}})} + A_{22}e^{i\Omega(t - \frac{x}{C_{22}})} \\ w_2^- &= B_{12}e^{i\Omega(t + \frac{x}{C_{12}})} + B_{22}e^{i\Omega(t - \frac{x}{C_{22}})} \end{aligned}\right\}, \qquad 0 < x < \ell$$

$$w_3^+ = A_{13}e^{i\Omega(t - \frac{x-\ell}{C_{13}})} + A_{23}e^{i\Omega(t - \frac{x-\ell}{C_{23}})}, \qquad \ell < x < \infty \tag{1.52}$$

where

$$C_{1j}^2 = \left(\frac{E_j I_j}{\rho_j A_j}\right)^{1/2}\Omega \quad , \quad C_{2j}^2 = -\left(\frac{E_j I_j}{\rho_j A_j}\right)^{1/2}\Omega = -C_{1j}^2 \quad , \quad j=1,2,3$$

The corresponding boundary conditions are

$$w_1(0,t) = w_2(0,t)$$

$$\frac{\partial w_1(0,t)}{\partial x} = \frac{\partial w_2(0,t)}{\partial x}$$

$$M_1(0,t) = M_2(0,t)$$

$$Q_1(0,t) = Q_2(0,t)$$

[illegible]

$$w_2(\ell,t) = w_3(\ell,t)$$

$$\frac{\partial w_2(\ell,t)}{\partial x} = \frac{\partial w_3(\ell,t)}{\partial x}$$

$$M_2(\ell,t) = M_3(\ell,t)$$

$$Q_2(\ell,t) = Q_3(\ell,t) \tag{1.53}$$

where

$$M_j = -E_j I_j \frac{\partial^2 w_j}{\partial x^2}, \qquad j = 1,2,3$$

$$Q_j = -E_j I_j \frac{\partial^3 w_j}{\partial x^3}, \qquad j = 1,2,3$$

and we obtain the following simultaneous, linear, algebraic equations (in matrix form):

$$\begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & -i & d & -id & -d & id & 0 & 0 \\ 1 & -1 & -\gamma d^2 & \gamma d^2 & -\gamma d^2 & \gamma d^2 & 0 & 0 \\ 1 & i & \gamma d^3 & i\gamma d^3 & -\gamma d^3 & -i\gamma d^3 & 0 & 0 \\ 0 & 0 & \varepsilon^{-i} & \varepsilon^{-1} & \varepsilon^{i} & \varepsilon & -1 & -1 \\ 0 & 0 & d\varepsilon^{-i} & -id\varepsilon^{-1} & -d\varepsilon^{i} & id\varepsilon & -1 & i \\ 0 & 0 & \gamma d^3\varepsilon^{-i} & -\gamma d^2\varepsilon^{-1} & \gamma d^2\varepsilon^{i} & -\gamma d^3\varepsilon & -1 & 1 \\ 0 & 0 & \gamma d^3\varepsilon^{-i} & i\gamma d^3\varepsilon^{-1} & -\gamma d^3\varepsilon^{i} & -i\gamma d^3\varepsilon & -1 & -i \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ A_{12} \\ A_{22} \\ B_{12} \\ B_{22} \\ A_{13} \\ A_{23} \end{bmatrix} = \begin{bmatrix} -A_{11} \\ A_{11} \\ -A_{11} \\ A_{11} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.54}$$

where

$$\gamma = \frac{E_2}{E_1} = \frac{E_2}{E_2}, \quad d = \frac{C_1}{C_2} = \frac{C_3}{C_2} = \left(\frac{\eta}{\gamma}\right)^{1/4}$$

$$\eta = \frac{\rho_2}{\rho_1} = \frac{\rho_2}{\rho_3}, \quad \varepsilon = e^{\frac{\omega}{C_{12}}\ell}$$

The transmission and reflection coefficients in this case are defined in a manner similar to equation (1.45). Two numerical results are also shown in Figures 1.8 and 1.9 for comparison purposes.

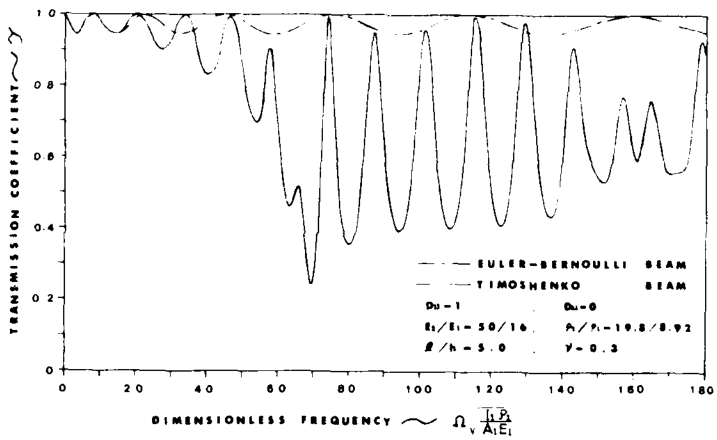

Figure 1.8 - Transmission Coefficient vs. Frequency, Copper-Tungsten-Copper Bonded Beams

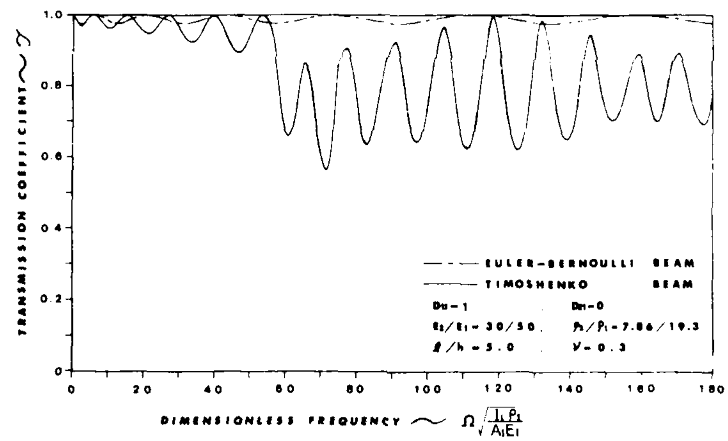

Figure 1.9 - Transmission Coefficient vs. Frequency, Tungsten-Iron-Tungsten Bonded Beams

4. CONCLUSION AND DISCUSSION OF RESULTS (PART I)

4.1 [illegible]

In the case of Euler-Bernoulli Beam Theory the medium is dispersive, two modes of propagation are possible, the first mode response is always a travelling sinusoid, and while the second mode response is always an attenuated standing wave. A study of Timoshenko Beam Theory also indicates the existence of two modes, the characteristics of which differ from that of foregoing modes. In response to harmonic excitation, the first mode response (lower phase velocity) will always be a travelling sinusoid, while the second mode response (higher phase velocity) will be in the form of a travelling sinusoid if $a < 1$ (high frequencies) or an attenuated standing wave if $a > 1$ (low frequencies). It can be shown that the mean energy flux vanishes in the case of a standing wave. The corresponding phase velocities are shown in Figure 1.1.

4.2 [illegible] Semi-Infinite Beams [illegible] of Different Materials

The incoming wave is assumed to be sinusoidal. With reference to Figures 1.5 through 1.6, we note that Timoshenko Beam Theory transmission coefficients T near zero frequency approach those of Euler-Bernoulli Beam Theory. In the low frequency range we have $a_1 > 1$, $a_2 > 1$, and therefore transmitted and reflected waves have (travelling) sinusoidal and (standing) attenuated components. In the high frequency range we have $a_1 < 1$, $a_2 < 1$, and in this case all wave components are sinusoidal. Large discrepancies between the two theories occur in two frequency bands, one of which separates the high from the low frequency region, and the other of which is the high frequency region. In the case depicted in Figures 1.5 and 1.6, that region is characterized by $a_1 < 1$ and $a_2 > 1$, i.e., reflected waves are sinusoidal, but transmitted waves are sinusoidal and attenuated (standing), resulting in a pronounced lowering of the transmission coefficient. In the cases exemplified by Figures 1.4 and 1.5, that region is characterized by $a_1 > 1$ and $a_2 < 1$, and in this case the transmitted waves are sinusoidal while the reflected waves are sinusoidal and attenuated. In the high frequency region transmission coefficients approach a constant value which is lower than those of Euler-Bernoulli Beam Theory as the frequencies approach infinity. It is also interesting to note (with reference to Figures 1.5 through 1.6) that in the region which is intermediate between the low and high frequency region the specific variations of the transmission coefficients with frequencies are strongly dependent on the direction of travel of the incoming wave. This is not the case for the Euler-Bernoulli Beam Theory where the transmission coefficient is independent of frequency as well as of the direction of travel of the incoming wave.

4.3 Beam of Finite Length Bonded to Two Semi-Infinite Beams Made of a Different Material

In this case we have both sinsuoidal and attenuated waves in all regions of the beam for the low frequency region ($a_1 > 1$, $a_2 > 1$) and only sinusoidal waves in all regions of the beam for the high frequency region ($a_1 < 1$, $a_2 < 1$). For the particular examples depicted in Figures 1.8 and 1.9, the transition region occurs when there are sinusoidal waves in domain 1 ($a_1 < 1$) and sinusoidal and attenuated waves in domain 2 ($a_2 > 1$) in Figure 1.2. In the low frequency region there is good to fair agreement between the two theories, while in the high frequency region there appears a difference between the two theories. In the case of Timoshenko Beam Theory, the numbers of maxima (and minima) of T per unit frequency approach a constant value as the frequencies increase, while in the case of Euler-Bernoulli Beam Theory, they decrease as the frequencies increase. These differences are due to the difference in mode properties and phase velocities between the two theories.

NOMENCLATURE - PART I

Quantity	Description
A	area of the cross-section of the beam
a	$A\kappa^2 G/I\omega^2$
C	phase velocity
d	$(n/\gamma)^{1/4}$
E	Young's modulus
e_{xy}	components of strain tensor
G	shearing modulus
h	thickness of the beam $= \sqrt{12\frac{I}{A}}$
I	moment of inertia of the beam
i	$\sqrt{-1}$
J	energy flux
j	domain of the beam
k	mode of the motion
ℓ	length of the beam
M	bending moment
m	applied surface force
n	ρ_2/ρ_1
Q,Q'	shearing force
q	applied surface force
$\mathcal{R}$	reflection coefficient
γ	$E_2/E_1 = G_2/G_1$
T	kinetic energy
$\mathcal{T}$	transmission coefficient

t	time
U_x, U_y, U_z	displacement components in the direction of x,y,z respectively
V	potential energy
v	width of the beam
x,y,z	rectangular coordinates
ν	Poisson's ratio
ρ	density
τ_{xx}	components of stress tensor
Ω	frequency

PART II

5. BASIC EQUATIONS OF PLATES

5.1 *Description of the Motion*

We shall assume the following displacement components for a plate:

$$
\begin{aligned}
U_x &= z\,\psi_x(x,y,t) \\
U_y &= z\,\psi_y(x,y,t) \\
U_z &= w\,(x,y,t) \quad .
\end{aligned}
\tag{2.1}
$$

The strain-displacement relationship can be written as follows:

$$
\begin{aligned}
e_{xx} &= \frac{\partial U_x}{\partial x} = z\,\frac{\partial \psi_x}{\partial x} \\
e_{yy} &= \frac{\partial U_y}{\partial y} = z\,\frac{\partial \psi_y}{\partial y} \\
e_{zz} &= \frac{\partial U_z}{\partial z} = 0 \\
e_{xy} &= \frac{1}{2}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) = \frac{1}{2}\cdot z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \\
e_{yz} &= \frac{1}{2}\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) = \frac{1}{2}\left(\psi_y + \frac{\partial w}{\partial y}\right) \\
e_{zx} &= \frac{1}{2}\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \psi_x\right) \quad .
\end{aligned}
\tag{2.2}
$$

5.2 *Energy Consideration and Hamilton's Principle*

In the theory of small motions of an elastic isotropic continuum, the kinetic and potential energies of deformation are

$$T = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^a \int_0^\ell \frac{1}{2}\rho\,(\dot{U}_x^2 + \dot{U}_y^2 + \dot{U}_z^2)\,dx\,dy\,dz$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^a \int_0^\ell \frac{1}{2}\rho\left\{z^2(\dot{\psi}_x^2 + \dot{\psi}_y^2) + \dot{w}^2\right\}dx\,dy\,dz$$

$$= \int_0^a \int_0^\ell \frac{1}{2}\rho\left\{\frac{h^3}{12}(\dot{\psi}_x^2 + \dot{\psi}_y^2) + h\dot{w}^2\right\}dx\,dy\,dz \quad , \tag{2.3}$$

$$V = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^a \int_0^\ell (\tau_{xx}e_{xx} + \tau_{yy}e_{yy} + \tau_{zz}e_{zz} + 2\tau_{xy}e_{xy} +$$

$$+ 2\tau_{yz}e_{yz} + 2\tau_{zx}e_{zx})\,dx\,dy\,dz$$

$$= \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^a \int_0^\ell \left\{\tau_{xx} z \frac{\partial\psi_x}{\partial x} + \tau_{yy} z \frac{\partial\psi_y}{\partial y} + \tau_{xy} z\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)\right.$$

$$\left. + \tau_{yz}\left(\psi_y + \frac{\partial w}{\partial y}\right) + \tau_{zx}\left(\frac{\partial w}{\partial x} + \psi_x\right)\right\}dx\,dy\,dz$$

$$= \frac{1}{2}\int_0^a \int_0^\ell \left\{M_x \frac{\partial\psi_x}{\partial x} + M_y \frac{\partial\psi_y}{\partial y} + M_{xy}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)\right.$$

$$\left. + Q_y\left(\psi_y + \frac{\partial w}{\partial y}\right) + Q_x\left(\frac{\partial w}{\partial x} + \psi_x\right)\right\}dx\,dy \quad , \tag{2.4}$$

where a dot indicates differentiation with respect to time, and the stress resultants are defined as follows:

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx}\, z\, dz$$

$$M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yy}\, z\, dz$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\, z\, dz$$

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx}\, dz$$

$$Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz}\, dz \quad . \qquad (2.5)$$

The variations of T and V are

$$\delta T = \int_0^a \int_0^\ell \rho \left\{ \frac{h^3}{12} (\dot{\psi}_x \delta\dot{\psi}_x + \dot{\psi}_y \delta\dot{\psi}_y) + h\dot{w}\delta\dot{w} \right\} dx\, dy \quad , \qquad (2.6)$$

$$\delta V = \int_0^a \int_0^\ell \left\{ M_x \frac{\partial \delta\psi_x}{\partial x} + M_y \frac{\partial \delta\psi_y}{\partial y} + M_{xy} \left(\frac{\partial \delta\psi_x}{\partial y} + \frac{\partial \delta\psi_y}{\partial x} \right) \right.$$
$$\left. + Q_y \left(\delta\psi_y + \frac{\partial \delta w}{\partial y} \right) + Q_x \left(\frac{\partial \delta w}{\partial x} + \delta\psi_y \right) \right\} dx\, dy \quad . \qquad (2.7)$$

Denote the work done by surface tractions when the displacements are varied by δW,

$$\delta W = \int_0^a \int_0^\ell (P\delta w + S_x \delta\psi_x + S_y \delta\psi_y) dx\, dy$$
$$+ \int_0^a [M_x^* \delta\psi_x + M_{xy}^* \delta\psi_y + Q_x^* \delta w]_{x=o}^{x=\ell}\, dy$$
$$+ \int_0^\ell [M_y^* \delta\psi_y + M_{xy}^* \delta\psi_x + Q_y^* \delta w]_{y=o}^{y=a}\, dx \quad , \qquad (2.8)$$

where

$$P = \tau_{zz}\big]_{z=\frac{h}{2}} - \tau_{zz}\big]_{z=-\frac{h}{2}}$$

$$S_x = \tau_{zx}\big]_{z=\frac{h}{2}} - \tau_{zx}\big]_{z=-\frac{h}{2}}$$

$$S_y = \tau_{zy}\big]_{z=\frac{h}{2}} - \tau_{zy}\big]_{z=-\frac{h}{2}}$$

$$M_x^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xx}\, z\, dz \quad , \quad \text{at} \quad x = 0 \quad \text{or} \quad x = \ell$$

$$M_y^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yy}\, z\, dz \quad , \quad \text{at} \quad y = 0 \quad \text{or} \quad x = a$$

$$M_{xy}^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy}\, z\, dz \quad , \quad \text{at} \quad x = 0 \quad \text{or} \quad x = \ell$$

$$M_{yx}^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx}\, z\, dz \quad , \quad \text{at} \quad y = 0 \quad \text{or} \quad x = a$$

$$Q_x^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz}\, dz \quad , \quad \text{at} \quad x = 0 \quad \text{or} \quad x = \ell$$

$$Q_y^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz}\, dz \quad , \quad \text{at} \quad y = 0 \quad \text{or} \quad x = a \quad .$$

Hamilton's principle in an interval of time t_1 to t_2 is

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = 0 \quad . \tag{2.9}$$

After substitution of equations (2.6), (2.7), and (2.8) into (2.9) and upon application of integration by parts, we are led to the following equation

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_0^a\int_0^\ell \Bigg\{ & \left(-\frac{\rho h^3}{12}\ddot{\psi}_x + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + S_x\right)\delta\psi_x \\
& + \left(-\frac{\rho h^3}{12}\ddot{\psi}_y + \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y + S_y\right)\delta\psi_y \\
& + \left(-\rho h\ddot{w} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P\right)\delta w\Bigg\}\, dx\, dy\, dt \qquad (2.10) \\
& + \int_{t_1}^{t_2}\int_0^a \left[(M_x^* - M_x)\delta\psi_x + (M_{xy}^* - M_{xy})\delta\psi_y + (Q_x^* - Q_x)\delta w\right]_{x=o}^{x=\ell} dy\, dt \\
& + \int_{t_1}^{t_2}\int_0^\ell \left[(M_y^* - M_y)\delta\psi_y + (M_{xy}^* - M_{xy})\delta\psi_x + (Q_y^* - Q_y)\delta w\right]_{y=o}^{y=a} dx\, dt = 0
\end{aligned}
$$

5.3 *Equations of the Motion*

From equation (2.10), the equations of motion can be written as follows:

$$
\begin{aligned}
& -\frac{\rho h^3}{12}\ddot{\psi}_x + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + S_x = 0 \\
& -\frac{\rho h^3}{12}\ddot{\psi}_y + \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y + S_y = 0 \\
& -\rho h\ddot{w} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P = 0 \qquad (2.11)
\end{aligned}
$$

The appropriate boundary conditions are

1) At $x=0$, $x=\ell$, one member of each of the pairs (M_x, ψ_x), (M_{xy}, ψ_y), and (Q_x, w) must be specified.

2) At $y=0$, $y=a$, one member of each of the pairs (M_y, ψ_y), $M_{xy}, \psi_x)$, and (Q_y, w) must be specified.

To insure a unique solution, the following initial conditions are required:

$\psi_x(x,y,0)$, $\dot{\psi}_x(x,y,0)$, $\psi_y(x,y,0)$, $\dot{\psi}_y(x,y,0)$, $w(x,y,0)$ and $\dot{w}(x,y,0)$ are specified.

The general Hooke's law reduces for the present isotropic three dimensional case to

$$\tau_{xx} = Ee_{xx} + \nu\,(\tau_{yy} + \tau_{zz})$$

$$\tau_{yy} = Ee_{yy} + \nu\,(\tau_{zz} + \tau_{xx})$$

$$\tau_{zz} = Ee_{zz} + \nu\,(\tau_{xx} + \tau_{yy}) \quad . \qquad (2.12)$$

$$\tau_{xy} = 2Ge_{xy}\ , \quad \tau_{yz} = 2Ge_{yz}\ , \quad \tau_{zx} = 2Ge_{zx} \quad . \qquad (2.13)$$

From the equations (2.12) we obtain

$$\tau_{xx} = \frac{E}{1-\nu^2}\,(e_{xx} + \nu\, e_{yy}) + \frac{\nu}{1-\nu}\,\tau_{zz}$$

$$\tau_{yy} = \frac{E}{1-\nu^2}\,(e_{yy} + \nu\, e_{xx}) + \frac{\nu}{1-\nu}\,\tau_{zz} \quad . \qquad (2.12')$$

Upon substitution of equations (2.12') into (2.5) we obtain

$$M_x = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} (ze_{xx}+\nu ze_{yy})dz + \frac{\nu}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\tau_{zz}\,dz$$

$$M_y = \frac{E}{1-\nu^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} (ze_{yy}+\nu ze_{xx})dz + \frac{\nu}{1-\nu}\int_{-\frac{h}{2}}^{\frac{h}{2}} z\,\tau_{zz}\,dz \ . \qquad (2.14)$$

Since the second terms of equations (2.14) are negligible in comparison with the first terms, we obtain

[illegible]

$$M_x = \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z e_{xx} + \nu z e_{yy})\, dz$$

$$M_y = \frac{E}{1-\nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z e_{yy} + \nu z e_{xx})\, dz \quad , \tag{2.15a}$$

and upon substitution of equation of (2.13) into (2.5) we obtain

$$M_{xy} = 2G \int_{-\frac{h}{2}}^{\frac{h}{2}} z e_{xy}\, dz$$

$$Q_x = 2G \int_{-\frac{h}{2}}^{\frac{h}{2}} e_{zx}\, dz$$

$$Q_y = 2G \int_{-\frac{h}{2}}^{\frac{h}{2}} e_{yz}\, dz \quad . \tag{2.15b}$$

Further substitution of equations (2.2) into (2.15a) and (2.15b) and using Mindlin's correction factor κ (4), the stress resultants and displacement relations become

$$M_x = D\left(\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\right)$$

$$M_y = D\left(\frac{\partial\psi_y}{\partial y} + \nu\frac{\partial\psi_x}{\partial x}\right)$$

$$M_{xy} = D\,\frac{1-\nu}{2}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)$$

$$Q_x = \kappa^2 Gh\left(\frac{\partial w}{\partial x} + \psi_x\right) \quad ,$$

$$Q_y = \kappa^2 Gh\left(\psi_y + \frac{\partial w}{\partial y}\right) \quad , \tag{2.16}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad , \quad G = \frac{E}{2(1+\nu)} \quad .$$

The displacement equations of motion are now obtained by substitution of (2.16) into (2.11), with the result:

$$D\left(\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^2\psi_y}{\partial x\partial y}\right) + D\,\frac{1-\nu}{2}\left(\frac{\partial^2\psi_x}{\partial y^2} + \frac{\partial^2\psi_y}{\partial x\partial y}\right) -$$

$$- \kappa^2 Gh\left(\frac{\partial w}{\partial x} + \psi_x\right) + S_x = \frac{\rho h^3}{12}\ddot{\psi}_x$$

$$D\left(\frac{\partial^2\psi_y}{\partial y^2} + \nu\frac{\partial^2\psi_x}{\partial x\partial y}\right) + D\,\frac{1-\nu}{2}\left(\frac{\partial^2\psi_x}{\partial x\partial y} + \frac{\partial^2\psi_y}{\partial x^2}\right) -$$

$$- \kappa^2 Gh\left(\psi_y + \frac{\partial w}{\partial y}\right) + S_y = \frac{\rho h^3}{12}\ddot{\psi}_y$$

$$\kappa^2 Gh\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial\psi_x}{\partial x}\right) + \kappa^2 Gh\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + P = \rho h\ddot{w} \quad . \qquad (2.17)$$

Equations (2.17) were obtained my Mindlin in reference [4].

5.4 *Energy Flux*

The sum of kinetic and potential energies for a plate of length $x_2 - x_1$ and width $y_2 - y_1$, with reference to equations (2.3) and (2.4), is

$$T + V = \frac{1}{2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left[\rho\left\{\frac{h^3}{12}\left(\dot{\psi}_x^2 + \dot{\psi}_y^2\right) + h\dot{w}^2\right\}\right.$$

$$+ M_x\frac{\partial\psi_x}{\partial x} + M_y\frac{\partial\psi_y}{\partial y} + M_{xy}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)$$

$$\left. + Q_x\left(\psi_y + \frac{\partial w}{\partial y}\right) + Q_y\left(\frac{\partial w}{\partial x} + \psi_x\right)\right] dx\, dy \quad . \qquad (2.18)$$

After differentiation of (2.18) with respect to time and upon application of integration by parts, we obtain

$$\frac{\partial}{\partial t}(T+V) = -\int_{y_1}^{y_2}\int_{x_1}^{x_2}\Bigg\{\dot{\psi}_x\left(-\rho\frac{h^3}{12}\ddot{\psi}_x + \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\right)$$

$$+ \dot{\psi}_y\left(-\frac{\rho h^3}{12}\ddot{\psi}_y + \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y\right)$$

$$+ \dot{w}\left(-\rho h\ddot{w} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)\Bigg\}\,dx\,dy$$

$$+\int_{y_1}^{y_2}\left[M_x\dot{\psi}_x + M_{xy}\dot{\psi}_y + Q_x\dot{w}\right]_{x_1}^{x_2} dy$$

$$+\int_{x_1}^{x_2}\left[M_y\dot{\psi}_y + M_{xy}\dot{\psi}_x + Q_y\dot{w}\right]_{y_1}^{y_2} dx \quad . \qquad (2.19)$$

When there are no applied loads, $S_x = S_y = P = 0$, the first integrand of the right hand side is identically zero. When the edges $y = y_1$ and $y = y_2$ of the plate are simply supported, the third integrand is identically zero. Thus the equation reduces to

$$\frac{\partial}{\partial t}(T-V) = \int_{y_1}^{y_2}\left[M_x\dot{\psi}_x + M_{xy}\dot{\psi}_y + Q_x\dot{w}\right]_{x_1}^{x_2} dy \quad , \qquad (2.20)$$

and we identify the energy flux as

$$J(x,t) = -\int_{y_1}^{y_2}\left[M_x\dot{\psi}_x + M_{xy}\dot{\psi}_y + Q_x\dot{w}\right]_{x=x} dy \quad . \qquad (2.21)$$

5.5 *Reduction to Elementary Theory*

If we let

$$\psi_x = -\frac{\partial w}{\partial x} \quad , \qquad \psi_y = -\frac{\partial w}{\partial y} \quad ,$$

in equation (2.10), we are led to the following equation

$$\int_{t_1}^{t_2}\int_0^a\int_0^{\ell}\left\{\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}-\rho h\ddot{w}\right.$$
$$\left.+\frac{\rho h^3}{12}\left(\frac{\partial^2 \ddot{w}}{\partial x^2}+\frac{\partial^2 \ddot{w}}{\partial y^2}\right)+\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+P\right\}\delta w\,dx\,dy\,dt$$

$$+\int_{t_1}^{t_2}\int_0^a\left[-\left(M_x^*-M_x\right)\frac{\partial\delta w}{\partial x}+\left\{\frac{\partial}{\partial y}\left(M_{xy}^*-M_{xy}\right)+\left(Q_x^*-Q_x\right)\right\}\delta w\right]_{x=o}^{x=\ell}dy\,dt$$

$$+\int_{t_1}^{t_2}\int_0^a\left[-\left(M_y^*-M_y\right)\frac{\partial\delta w}{\partial y}+\left\{\frac{\partial}{\partial x}\left(M_{xy}^*-M_{xy}\right)+\left(Q_y^*-Q_y\right)\right\}\delta w\right]_{y=o}^{y=a}dx\,dt$$

$$-\int_{t_1}^{t_2}\int_0^a\left[\left(\frac{\rho h^3}{12}\frac{\partial\ddot{w}}{\partial x}+\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x+S_x\right)\delta w\right]_{x=o}^{x=\ell}dy\,dt$$

$$-\int_{t_1}^{t_2}\int_0^{\ell}\left[\left(\frac{\rho h^3}{12}\frac{\partial\ddot{w}}{\partial y}+\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y+S_y\right)\delta w\right]_{y=o}^{y=a}dx\,dt$$

$$-2\int_{t_1}^{t_2}\left[\left[\left(M_{xy}^*-M_{xy}\right)\delta w\right]_{x=o}^{x=\ell}\right]_{y=o}^{y=a}dt=0\quad. \tag{2.22}$$

Thus, the equation of motion can be written as follows:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+P$$
$$+\frac{\rho h^3}{12}\left(\frac{\partial^2 \ddot{w}}{\partial x^2}+\frac{\partial^2 \ddot{w}}{\partial y^2}\right)+\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}=\rho h\ddot{w}\quad. \tag{2.23}$$

If we delete

$$\frac{\rho h^3}{12}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right),\quad\frac{\partial S_x}{\partial x}\quad\text{and}\quad\frac{\partial S_y}{\partial y}\quad,$$

we obtain

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+P=\rho h\ddot{w}\quad. \tag{2.24}$$

From the first and second equations of (2.11) we obtain

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + S_x - \frac{\rho h^3}{12} \cdot \frac{\partial \ddot{w}}{\partial x}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + S_y - \frac{\rho h^3}{12} \cdot \frac{\partial \ddot{w}}{\partial y} \quad . \qquad (2.25)$$

If we delete

$$\frac{\rho h^3}{12} \cdot \frac{\partial \ddot{w}}{\partial x} \; , \; \frac{\rho h^3}{12} \cdot \frac{\partial \ddot{w}}{\partial y} \; , \; S_x \quad \text{and} \quad S_y \quad ,$$

we obtain

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad . \qquad (2.26)$$

Since the fourth and fifth terms vanish because of the equations (2.11), the appropriate boundary conditions are as follows:

(1) At $x = 0$, $x =$ [illegible], one member of each of the pairs

$$\left(M_x , \frac{\partial w}{\partial y} \right) \; , \; \left(\frac{\partial M_{xy}}{\partial x} + Q_x , w \right) \quad ,$$

must be specified.

(2) At $y = 0$, $y = a$, one member of each of the pairs

$$\left(M_y , \frac{\partial w}{\partial y} \right) \; , \; \left(\frac{\partial M_{xy}}{\partial x} + Q_y , w \right) \quad ,$$

must be specified.

(3) At any two points of four corners $(0,0)$, $(0,\ell)$, (a,o) and (a,ℓ), one member of the pair (M_{xy}, w) must be specified.

The energy flux equation in this case reduces to

$$J(x,t) = - \int_{y_1}^{y_2} \left[- M_x \frac{\partial \dot{w}}{\partial x} - \left(\frac{\partial M_{xy}}{\partial y} - Q_x \right) \dot{w} \right]_{x=x} dy \quad . \qquad (2.27)$$

The displacement equations of motion and the stress resultants are obtained as follows:

$$-D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + P = \rho h\ddot{w} \quad . \tag{2.28}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \quad . \tag{2.29}$$

6. WAVE PROPAGATION IN PIECEWISE NON-HOMOGENEOUS PLATES

6.1 *General Solution*

Equations (2.17) with $S_x = S_y = P = 0$ reduce to

$$D\left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) + D\frac{1-\nu}{2}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) -$$

$$- \kappa^2 Gh\left(\frac{\partial w}{\partial x} + \psi_x\right) = \frac{\rho h^3}{12}\ddot{\psi}_x$$

$$\kappa^2 Gh\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) = \rho h\ddot{w}$$

$$D\left(\frac{\partial^2 \psi_y}{\partial y^2} + \nu\frac{\partial^2 \psi_x}{\partial x \partial y}\right) + D\frac{1-\nu}{2}\left(\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial x^2}\right) -$$

$$- \kappa^2 Gh\left(\frac{\partial w}{\partial y} + \psi_y\right) = \frac{\rho h^3}{12}\ddot{\psi}_y \quad . \tag{2.30}$$

Since we consider the infinite plate simply supported at the edges, $y = 0$ and $y = a$, we assume

$$\psi_x(x,y,t) = A\, e^{i\Omega(t - \frac{x}{C})} \cdot \sin \frac{n\pi}{a} y$$

$$\psi_y(x,y,t) = B\, e^{i\Omega(t - \frac{x}{C})} \cdot \cos \frac{n\pi}{a} y$$

$$w(x,y,t) = R\, e^{i\Omega(t - \frac{x}{C})} \cdot \sin \frac{n\pi}{a} y \quad , \qquad (2.31)$$

where Ω is wave frequency, and C is phase velocity. Upon substitution of equation (2.31) into (2.30), we obtain three homogeneous linear algebraic equations in A, B, and R, whose determinant, set equal to zero, yields the velocity equation;

$$\{\xi^4 - (\xi_1^2 + \xi_2^2)\xi^2 + \xi_1^2\xi_2^2\}(\xi^2 - \xi_3^2) = 0 \quad , \qquad (2.32)$$

where

$$\xi^2 = \left(\frac{\Omega}{C}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

$$\xi_1^2 - \xi_2^2 = \Omega^2\left(\frac{\rho}{\kappa^2 G} + \frac{h^3\rho}{12D}\right)$$

$$\xi_1^2\xi_2^2 = \Omega^2 \frac{\rho}{\kappa^2 G}\left(\Omega^2 \frac{h^3\rho}{12D} - \frac{\kappa^2 Gh}{D}\right) \quad , \qquad (2.33)$$

hence

$$\xi_1^2 = \frac{1}{2}\Omega^2\left\{\left(\frac{\rho}{\kappa^2 G} + \frac{h^3\rho}{12D}\right) - \sqrt{\left(\frac{\rho}{\kappa^2 G} - \frac{h^3\rho}{12D}\right)^2 + 4\frac{\rho h}{D\Omega^2}}\right\}$$

$$\xi_2^2 = \frac{1}{2}\Omega^2\left\{\left(\frac{\rho}{\kappa^2 G} + \frac{h^3\rho}{12D}\right) + \sqrt{\left(\frac{\rho}{\kappa^2 G} - \frac{h^3\rho}{12D}\right)^2 + 4\frac{\rho h}{D\Omega^2}}\right\}$$

$$\xi_3^2 = \frac{2}{1-\nu}\Omega^2\left\{\frac{h^3\rho}{12D} - \frac{\kappa^2 Gh}{\Omega^2 D}\right\} \qquad (2.34)$$

From equations (2.33) and (2.34) we obtain the three modes of velocity as follows:

$$C_k^2 = \frac{\Omega^2}{\xi_k^2 - \left(\frac{n\pi}{a}\right)^2} \qquad k = 1,2,3 \quad . \tag{2.35}$$

The most general solution of (2.30) is therefore

$$\psi_x(x,y,t) = \sum_{k=1}^{3} \left\{ A_k^+ \, e^{i\Omega(t - \frac{x}{C_k})} + A_k^- \, e^{i\Omega(t + \frac{x}{C_k})} \right\} \sin \frac{n\pi}{a} y$$

$$\psi_y(x,y,t) = \sum_{k=1}^{3} \left\{ B_k^+ \, e^{i\Omega(t - \frac{x}{C_k})} + B_k^- \, e^{i\Omega(t + \frac{x}{C_k})} \right\} \cos \frac{n\pi}{a} y$$

$$w(x,y,t) = \sum_{k=1}^{3} \left\{ R_k^+ \, e^{i\Omega(t - \frac{x}{C_k})} + R_k^- \, e^{i\Omega(t + \frac{x}{C_k})} \right\} \sin \frac{n\pi}{a} y \ , \tag{2.36}$$

where it is understood that the frequencies must be the same for all waves and that C_k can become imaginary. A plot of phase velocity vs. frequency is shown in Figure 2.1.

Note that of the 18 coefficients appearing in equation (2.36), only 6 are independent. After substitution of equations (2.36) into (2.30), we find that

$$B_k^+ = p_k \, A_k^+ \, i \quad , \qquad B_k^- = - \, p_k \, A_k^- \, i$$

$$R_k^+ = q_k \, A_k^+ \, i \quad , \qquad R_k^- = - \, q_k \, A_k^- \, i \quad , \tag{2.37}$$

where

$$p_k = \frac{C_k}{\Omega} \cdot \frac{n\pi}{a} \quad , \qquad k = 1,2, \qquad p_3 = - \frac{\Omega}{C_3} \cdot \frac{a}{n\pi}$$

$$q_k = - \frac{C_k}{\Omega} \cdot \frac{\xi_k^2}{\xi_k^2 - \Omega^2 \frac{\Omega}{\kappa^2 G}} \ , \ k = 1,2, \qquad q_3 = 0 \quad .$$

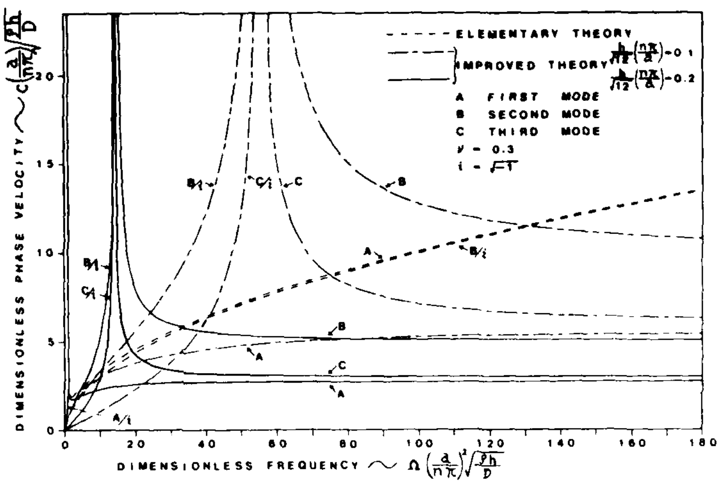

Figure 2.1 - Phase Velocity vs. Frequency

In elementary theory, we have

$$w(x,y,t) = \sum_{k=2}^{2} \left\{ R_k^+ e^{i\Omega(t - \frac{x}{C_k})} + R_k^- e^{i\Omega(t + \frac{x}{C_k})} \right\} \sin \frac{n\pi}{a} y \,, \tag{2.38}$$

where

$$C_1^2 = \frac{\Omega^2}{\Omega\sqrt{\frac{\rho h}{D}} - \left(\frac{n\pi}{a}\right)^2}$$

$$C_2^2 = \frac{\Omega^2}{-\Omega\sqrt{\frac{\rho h}{D}} - \left(\frac{n\pi}{a}\right)^2} \,.$$

Note that we can also obtain phase velocities as $h \to 0$ in the equation (2.32) of improved theory.

6.2 *Wave Motion in Two Bonded Semi-Infinite Plates Composed of Different Materials*

Two semi-infinite plates of different materials are bonded at x=0, as shown in Figure 2.2. For waves coming from the negative x direction, let ψ^+_{x1}, ψ^+_{y1}, and w^+_1 be the incoming waves, ψ^-_{x1}, ψ^-_{y1} and w^-_1 be the reflected waves, and let ψ^+_{x2}, ψ^+_{y2} and w^+_2 be the transmitted waves, we have

$$\left.\begin{aligned}
\psi^+_{x1}(x,y,t) &= \sum_{k=1}^{3} A^+_{k1}\, e^{i\Omega(t - \frac{x}{C_{k1}})} \sin\frac{n\pi}{a} y \\
\psi^+_{y1}(x,y,t) &= \sum_{k=1}^{3} B^+_{k1}\, e^{i\Omega(t - \frac{x}{C_{k1}})} \cos\frac{n\pi}{a} y \\
w^+_1(x,y,t) &= \sum_{k=1}^{3} R^+_{k1}\, e^{i\Omega(t - \frac{x}{C_{k1}})} \sin\frac{n\pi}{a} y \qquad (2.39a)\\
\psi^-_{x1}(x,y,t) &= \sum_{k=1}^{3} A^-_{k1}\, e^{i\Omega(t + \frac{x}{C_{k1}})} \sin\frac{n\pi}{a} y \\
\psi^-_{y1}(x,y,t) &= \sum_{k=1}^{3} B^-_{k1}\, e^{i\Omega(t + \frac{x}{C_{k1}})} \cos\frac{n\pi}{a} y \\
w^-_1(x,y,t) &= \sum_{k=1}^{3} R^-_{k1}\, e^{i\Omega(t + \frac{x}{C_{k1}})} \sin\frac{n\pi}{a} y \qquad (2.39b)
\end{aligned}\right\} -\infty < x < 0$$

$$\left.\begin{aligned}
\psi^+_{x2}(x,y,t) &= \sum_{k=1}^{3} A^+_{k2}\, e^{i\Omega(t - \frac{x}{C_{k2}})} \sin\frac{n\pi}{a} y \\
\psi^+_{y2}(x,y,t) &= \sum_{k=1}^{3} B^+_{k2}\, e^{i\Omega(t - \frac{x}{C_{k2}})} \cos\frac{n\pi}{a} y \\
w^+_2(x,y,t) &= \sum_{k=1}^{3} R^+_{k2}\, e^{i\Omega(t - \frac{x}{C_{k2}})} \sin\frac{n\pi}{a} y \qquad (2.39c)
\end{aligned}\right\} 0 < x < \infty$$

where

$$B^{+}_{kj} = p_{kj}\, A^{+}_{kj}\, i \quad , \qquad B^{-}_{kj} = -\, p_{kj}\, A^{-}_{kj}\, i$$

$$R^{+}_{kj} = q_{kj}\, A^{+}_{kj}\, i \quad , \qquad R^{-}_{kj} = -\, q_{kj}\, A^{-}_{kj}\, i$$

$$p_{kj} = \frac{C_{kj}}{\Omega}\, \frac{n\pi}{a} \quad , \quad k = 1,2 \quad , \qquad p_{3j} = -\frac{\Omega}{C_{3j}}\, \frac{a}{n\pi}$$

$$q_{kj} = -\frac{C_{kj}}{\Omega}\, \frac{\xi^{3}_{kj}}{\xi^{2}_{kj} - \Omega^{2} \dfrac{\rho_j}{\kappa^{2} G_j}} \quad , \quad k = 1,2 \quad , \qquad q_{3j} = 0$$

and where the subscript j = 1,2 refers to the respective domain of the plate (see Figure 2.2). The first subscript k = 1,2,3 refers to the mode of motion and the second subscript refers to the domain of the plate.

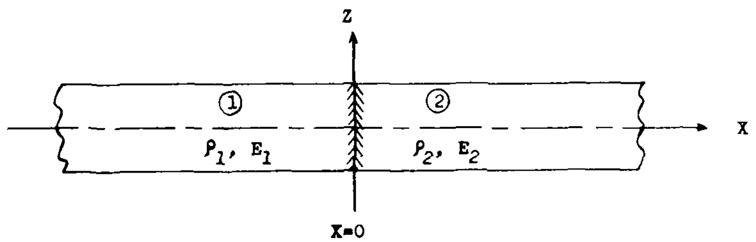

Figure 2.2 - Two Bonded, Semi-Infinite Plates

At the junction, x=0, the following six boundary conditions must be satisfied:

$$\begin{aligned}
\psi_{x1}(0,y,t) &= \psi_{x2}(0,y,t) \\
\psi_{y1}(0,y,t) &= \psi_{y2}(0,y,t) \\
w_{1}(0,y,t) &= w_{2}(0,y,t) \\
M_{x1}(0,y,t) &= M_{x2}(0,y,t) \\
M_{xy1}(0,y,t) &= M_{xy2}(0,y,t) \\
Q_{x1}(0,y,t) &= Q_{x2}(0,y,t) \quad ,
\end{aligned} \qquad (2.40)$$

where

$$\psi_{x1} = \psi_{x1}^{+} + \psi_{x1}^{-}$$

$$\psi_{y1} = \psi_{y1}^{+} + \psi_{y1}^{-}$$

$$w_1 = w_1^{+} + w_1^{-}$$

$$\psi_{x2} = \psi_{x2}^{+}$$

$$\psi_{y2} = \psi_{y2}^{+}$$

$$w_2 = w_2^{+}$$

$$M_{xj} = D_j\left(\frac{\partial\psi_{xj}}{\partial x} + \nu\frac{\partial\psi_{yj}}{\partial y}\right) \quad , \qquad j = 1,2$$

$$M_{xyj} = D_j\,\frac{1-\nu}{2}\left(\frac{\partial\psi_{xj}}{\partial y} + \frac{\partial\psi_{yj}}{\partial x}\right) \quad , \qquad j = 1,2$$

$$Q_{xj} = \kappa^2 G_j h\left(\frac{\partial w_j}{\partial x} + \psi_{xj}\right) \quad , \qquad j = 1,2$$

Upon substitution of equations (2.39) into (2.40), we obtain a set of simultaneous, linear, algebraic, equations (in matrix form):

$$\begin{bmatrix} [\ 1\ ] & [\ -1\ ] \\ [\ p_{k1}\ ] & [\ p_{k2}\ ] \\ [\ q_{k1}\ ] & [\ q_{k2}\ ] \\ [\ \gamma_{k1}\ ] & [\ m\gamma_{k2}\ ] \\ [\ f_{k1}\ ] & [-mf_{k2}] \\ [\ g_{k1}\ ] & [-mg_{k2}] \end{bmatrix} \begin{bmatrix} A_{11}^{-} \\ A_{21}^{-} \\ A_{31}^{-} \\ A_{12}^{+} \\ A_{22}^{+} \\ A_{32}^{+} \end{bmatrix} = \begin{bmatrix} [\ -A_{k1}^{+}\ ] \\ [p_{k1}A_{k1}^{+}\ ] \\ [q_{k1}A_{k1}^{+}\ ] \\ [\ \gamma_{k1}A_{k1}^{+}] \\ [-f_{k1}A_{k1}^{+}] \\ [-g_{k1}A_{k1}^{+}] \end{bmatrix} \quad , \qquad (2.41)$$

where, for example

$$\begin{bmatrix} [\ p_{k1}\ ] \\ \\ \end{bmatrix} \begin{bmatrix} A_{11}^- \\ A_{21}^- \\ A_{31}^- \end{bmatrix} = \sum_{k=1}^{3} p_{k1} A_{k1}^-$$

$$\begin{bmatrix} [p_{k1} A_{k1}^+] \\ \\ \end{bmatrix} \equiv \sum_{k=1}^{3} p_{k1} A_{k1}^+ \quad ,$$

and

$$m = \frac{E_2}{E_1} = \frac{G_2}{G_1}$$

$$\gamma_{kj} = \frac{\Omega}{C_{kj}} + \nu \frac{n\pi}{a} p_{kj} \quad , \quad k=1,2,3 \quad , \quad j=1,2$$

$$f_{kj} = \frac{n\pi}{a} + \frac{\Omega}{C_{kj}} p_{kj} \quad , \quad k=1,2,3 \quad , \quad j=1,2$$

$$g_{kj} = \frac{\Omega}{C_{kj}} q_{kj} + 1 \quad , \quad k=1,2,3 \quad , \quad j=1,2 \quad .$$

Once A_{11}^+, A_{21}^+, A_{31}^+ and Ω of the incoming waves and the material properties are known, the quantities A_{11}^-, A_{21}^-, A_{31}^-, A_{12}^+, A_{22}^+ and A_{23}^+ are obtained by the application of Cramer's Rule. When C_{11}, C_{21} or C_{31} are imaginary, A_{11}^+, A_{21}^+ or A_{31}^+ must be equal to zero, respectively, to insure a bounded solution for $x \to -\infty$.

The energy flux of the incoming wave, reflected wave, and transmitted wave is defined by equation (2.21), except that only real parts of (2.39) are used:

$$J_1^+ = -\int_{y_1}^{y_2} [(\mathrm{Re} M_{x1}^+)(\mathrm{Re}\dot{\psi}_{x1}^+) + (\mathrm{Re} M_{xy1}^+)(\mathrm{Re}\dot{\psi}_{y1}^+) + (\mathrm{Re} Q_{x1}^+)(\mathrm{Re}\dot{w}_1^+)]_x \, dy$$

$$J_1^- = -\int_{y_1}^{y_2} [(\mathrm{Re} M_{x1}^-)(\mathrm{Re}\dot{\psi}_{x1}^-) + (\mathrm{Re} M_{xy1}^-)(\mathrm{Re}\dot{\psi}_{y1}^-) + (\mathrm{Re} Q_{x1}^-)(\mathrm{Re}\dot{w}_1^-)]_x \, dy$$

$$J_2^+ = -\int_{y_1}^{y_2} [(\mathrm{Re} M_{x2}^+)(\mathrm{Re}\dot{\psi}_{x2}^+) + (\mathrm{Re} M_{xy2}^+)(\mathrm{Re}\dot{\psi}_{y2}^+) + (\mathrm{Re} Q_{x2}^+)(\mathrm{Re}\dot{w}_2^+)]_x \, dy \quad .$$

(2.42)

... transmission and reflection coefficients are therefore obtained in the following manner:

$$T = \frac{\overline{J_2^+(0,t)}}{\overline{J_1^+(0,t)}}$$

$$R \equiv \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}} \quad , \tag{2.43}$$

where the bar denotes the time average over a complete period.

From the point of view of conservation of energy, the sum of the transmission and reflection coefficients must be equal to one, i.e., $T + R = 1$. Calculated values of transmission coefficients vs. dimensionless frequency

$$\Omega \sqrt{\frac{\rho_1 h}{D_1}} \left(\frac{a}{n\pi}\right)^2 \quad ,$$

are plotted in Figures 2.3 through 2.6.

In elementary theory, the displacement equations become

$$\left.\begin{aligned} w_1^+(x,y.t) &= \sum_{k=1}^{2} R_{k1}^+ \, e^{i\Omega\left(t - \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y \\ w_1^-(x,y,t) &= \sum_{k=1}^{2} R_{k1}^- \, e^{i\Omega\left(t + \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y \end{aligned}\right\} , \quad -\infty < x < 0$$

$$w_2^+(x,y,t) = \sum_{k=1}^{2} R_{k2}^+ \, e^{i\Omega\left(t - \frac{x}{C_{k2}}\right)} \sin \frac{n\pi}{a} y \quad , \quad 0 < x < \infty \tag{2.44}$$

where

$$C_{1j}^2 = \frac{\Omega^2}{\Omega\sqrt{\frac{\rho_j h}{D_j}} - \left(\frac{n\pi}{a}\right)^2} \quad , \quad C_{2j}^2 = \frac{\Omega^2}{-\Omega\sqrt{\frac{\rho_j h}{D_j}} - \left(\frac{n\pi}{a}\right)^2} \quad .$$

At the junction, x = 0, the boundary conditions

$$w_1(0,y,t) = w_2(0,y,t)$$

$$\frac{\partial w_1(0,y,t)}{\partial x} = \frac{\partial w_2(0,y,t)}{\partial x}$$

$$M_{x1}(0,y,t) = M_{x2}(0,y,t)$$

$$\left[Q_{x1} + \frac{\partial M_{xy1}}{\partial y}\right]_{x=0} = \left[Q_{x2} + \frac{\partial M_{xy2}}{\partial y}\right]_{x=0} \quad , \qquad (2.45)$$

yield the following simultaneous, linear algebraic equations

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \frac{1}{C_{11}} & \frac{1}{C_{21}} & -\frac{1}{C_{12}} & -\frac{1}{C_{22}} \\ \rho_{11} & \rho_{21} & -m\rho_{12} & -m\rho_{22} \\ \eta_{11} & \eta_{21} & -m\eta_{12} & -m\eta_{22} \end{bmatrix} \begin{bmatrix} R_{11}^- \\ R_{21}^- \\ R_{12}^+ \\ R_{22}^+ \end{bmatrix} = \begin{bmatrix} -R_{11}^+ \\ \frac{1}{C_{11}} R_{11}^+ \\ \rho_{11} R_{11}^+ \\ \eta_{11} R_{11}^+ \end{bmatrix} \quad . \qquad (2.46)$$

where $A_{21}^+ = 0$ because C_{21} is always imaginary, and

$$\rho_{kj} = \left(\frac{\Omega}{C_{kj}}\right)^2 + \nu\left(\frac{n\pi}{a}\right)^2 \quad , \quad \eta_{kj} = \frac{1}{C_{kj}}\left\{\left(\frac{\Omega}{C_{kj}}\right)^2 + (2-\nu)\left(\frac{n\pi}{a}\right)^2\right\}$$

$$k=1,2, \qquad j=1,2 \quad .$$

The energy flux of the incoming wave, reflected wave, and transmitted wave is defined by equation (2.27) except that only real parts of (2.44) are used:

$$J_1^+ = -\int_{y_1}^{y_2}\left[-(\mathrm{Re}\ M_{x1}^+)\left(\mathrm{Re}\ \frac{\partial \dot{w}_1^+}{\partial x}\right) + \left\{\mathrm{Re}\left(\frac{\partial M_{xy1}^+}{\partial y} + Q_{x1}^+\right)\right\}(\mathrm{Re}\dot{w}_1^+)\right]_x dy$$

$$J^- = +\int_{y_1}^{y_2}\left[-(\mathrm{Re}\ M_{x1}^-)\left(\mathrm{Re}\ \frac{\partial \dot{w}_1^-}{\partial x}\right) + \left\{\mathrm{Re}\left(\frac{\partial M_{xy1}^-}{\partial y} + Q_{x1}^-\right)\right\}(\mathrm{Re}\dot{w}_1^-)\right]_x dy$$

$$J^+ = -\int_{y_1}^{y_2}\left[-(\mathrm{Re}\ M_{x2}^+)\left(\mathrm{Re}\ \frac{\partial \dot{w}_2^+}{\partial x}\right) + \left\{\mathrm{Re}\left(\frac{\partial M_{xy2}^+}{\partial y} + Q_{x2}^+\right)\right\}(\mathrm{Re}\dot{w}_2^+)\right]_x dy \quad . \qquad (2.47)$$

The transmission and reflection coefficients are

$$T = \frac{\overline{J_2^+(0,t)}}{\overline{J_1^+(0,t)}}$$

$$R \equiv \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}}$$

Numerical values of the transmission coefficients for the elementary theory are shown in Figures 2.3 through 2.6 for comparison purposes.

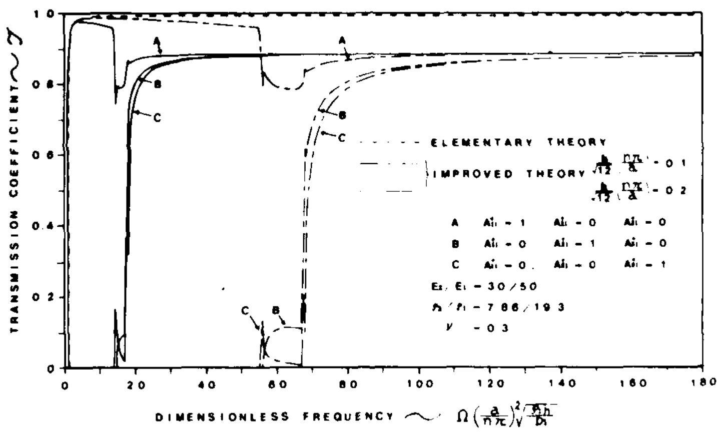

Figure 2.3 - Transmission Coefficient vs. Frequency, Tungsten-Iron Bonded Plates

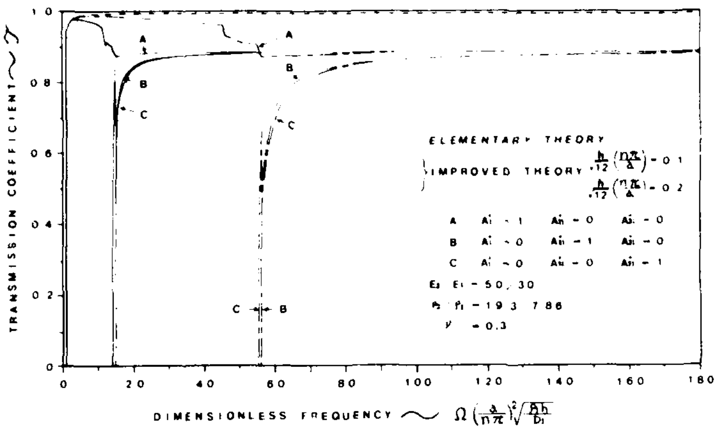

Figure 2.4 - Transmission Coefficient vs. Frequency, Iron-Tungsten Bonded Plates

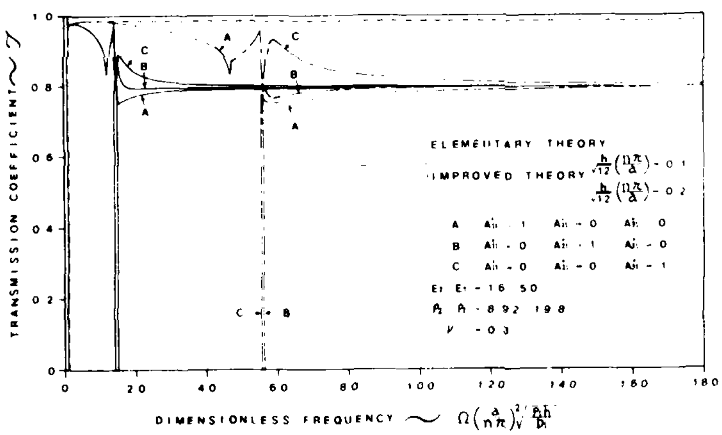

Figure 2.5 - Transmission Coefficient vs. Frequency, Tungsten-Copper Bonded Plates

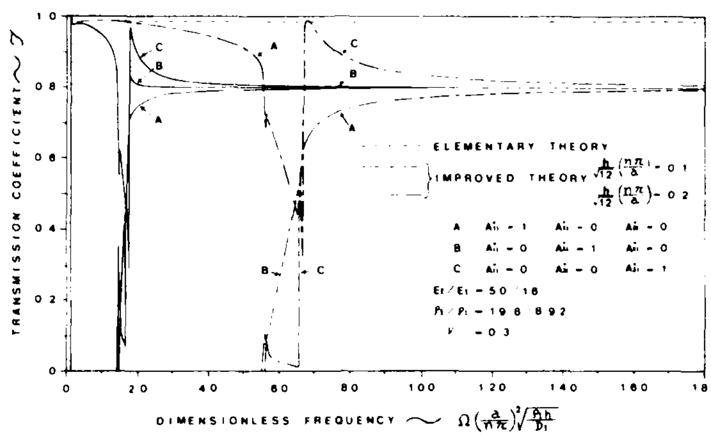

Figure 2.6 - Transmission Coefficient vs. Frequency, Copper-Tungsten Bonded Plates

6.3 *Wave Motion in a Plate of Finite Length Bonded at Each End to Two Semi-Infinite Plates Composed of a Different Material*

A plate of finite length is bonded at each end to two semi-infinite plates composed of a different material as shown in Figure 2.7.

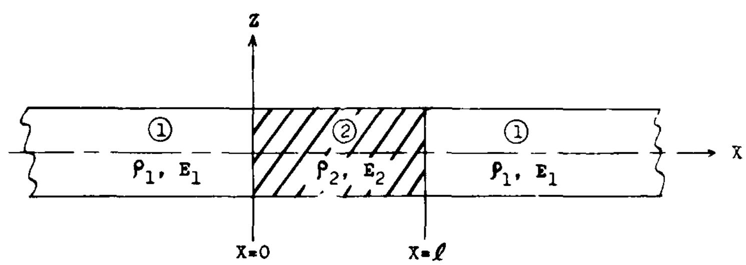

Figure 2.7 - Plate of Finite Length Bonded to Two Semi-Infinite Plates Composed of a Different Material

The displacement equations in this case are:

$$\left.\begin{aligned}
\psi_{x1}^{+}(x,y,t) &= \sum_{k=1}^{3} A_{k1}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y \\
\psi_{y1}^{+}(x,y,t) &= \sum_{k=1}^{3} B_{k1}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k1}}\right)} \cos \frac{n\pi}{a} y \\
w_{1}^{+}(x,y,t) &= \sum_{k=1}^{3} R_{k1}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y \\
\psi_{x1}^{-}(x,y,t) &= \sum_{k=1}^{3} A_{k1}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y \\
\psi_{y1}^{-}(x,y,t) &= \sum_{k=1}^{3} B_{k1}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k1}}\right)} \cos \frac{n\pi}{a} y \\
w_{1}^{-}(x,y,t) &= \sum_{k=1}^{3} R_{k1}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k1}}\right)} \sin \frac{n\pi}{a} y
\end{aligned}\right\}, \quad -\infty<x<0 \tag{2.48}$$

$$\left.\begin{aligned}
\psi_{x2}^{+}(x,y,t) &= \sum_{k=1}^{3} A_{k2}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k2}}\right)} \sin \frac{n\pi}{a} y \\
\psi_{y2}^{+}(x,y,t) &= \sum_{k=1}^{3} B_{k2}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k2}}\right)} \cos \frac{n\pi}{a} y \\
w_{2}^{+}(x,y,t) &= \sum_{k=1}^{3} R_{k2}^{+}\, e^{i\omega\left(t - \frac{x}{C_{k2}}\right)} \sin \frac{n\pi}{a} y
\end{aligned}\right\}, \quad 0<x<\ell$$

$$\left.\begin{aligned}
\psi_{x2}^{-}(x,y,t) &= \sum_{k=1}^{3} A_{k2}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k2}}\right)} \sin \frac{n\pi}{a} y \\
\psi_{y2}^{-}(x,y,t) &= \sum_{k=1}^{3} B_{k2}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k2}}\right)} \cos \frac{n\pi}{a} y \\
w_{2}^{-}(x,y,t) &= \sum_{k=1}^{3} R_{k2}^{-}\, e^{i\omega\left(t + \frac{x}{C_{k2}}\right)} \sin \frac{n\pi}{a} y
\end{aligned}\right\}, \quad 0<x<\ell$$

$$\left.\begin{aligned}
\psi^+_{x3}(x,y,t) &= \sum_{k=1}^{3} A^+_{k3}\, e^{i\Omega(t - \frac{x-\ell}{C_{k3}})} \sin \frac{n\pi}{a} y \\
\psi^+_{y3}(x,y,t) &= \sum_{k=1}^{3} B^+_{k3}\, e^{i\Omega(t - \frac{x-\ell}{C_{k3}})} \cos \frac{n\pi}{a} y \\
w^+_3(x,y,t) &= \sum_{k=1}^{3} R^+_{k3}\, e^{i\Omega(t - \frac{x-\ell}{C_{k3}})} \sin \frac{n\pi}{a} y
\end{aligned}\right\} , \quad \ell<x<\infty \tag{2.48}$$

The relations among the 45 coefficients in equations (2.48) are defined in a manner similar to those of equation (2.39). Note that $C_{k3} = C_{k1}$, $p_{k3} = p_{k1}$, $q_{k3} = q_{k1}$, $r_{k3} = r_{k1}$, $f_{k3} = f_{k1}$ and $g_{k3} = g_{k1}$ as k = 1,2,3, for this special case.

At the junction points x = 0, and x = ℓ, the following boundary conditions must be satisfied:

$$\begin{aligned}
\psi_{x1}(0,y,t) &= \psi_{x2}(0,y,t) \\
\psi_{y1}(0,y,t) &= \psi_{y2}(0,y,t) \\
w_1(0,y,t) &= w_2(0,y,t) \\
M_{x1}(0,y,t) &= M_{x2}(0,y,t) \\
M_{xy1}(0,y,t) &= M_{xy2}(0,y,t) \\
Q_{x1}(0,y,t) &= Q_{x2}(0,y,t) \\
\psi_{x2}(\ell,y,t) &= \psi_{x3}(\ell,y,t) \\
\psi_{y2}(\ell,y,t) &= \psi_{y3}(\ell,y,t) \\
w_2(\ell,y,t) &= w_3(\ell,y,t) \\
M_{x2}(\ell,y,t) &= M_{x3}(\ell,y,t) \\
M_{xy2}(\ell,y,t) &= M_{xy3}(\ell,y,t) \\
Q_{x2}(\ell,y,t) &= Q_{x3}(\ell,y,t)
\end{aligned} \tag{2.49}$$

where

$$
\begin{aligned}
\psi_{x1} &= \psi_{x1}^{+} + \psi_{x1}^{-} \\
\psi_{y1} &= \psi_{y1}^{+} + \psi_{y1}^{-} \\
w_1 &= w_1^{+} + w_1^{-} \\
\psi_{x2} &= \psi_{x2}^{+} + \psi_{x2}^{-} \\
\psi_{y2} &= \psi_{y2}^{+} + \psi_{y2}^{-} \\
w_2 &= w_2^{+} + w_2^{-} \\
\psi_{x3} &= \psi_{x3}^{+} \\
\psi_{y3} &= \psi_{y3}^{+} \\
w_3 &= w_3^{+}
\end{aligned}
$$

$$M_{xj} = D_j \left(\frac{\partial \psi_{xj}}{\partial x} + \nu \frac{\partial \psi_{xj}}{\partial y} \right) , \qquad j = 1,2,3$$

$$M_{xyj} = D_j \frac{1-\nu}{2} \left(\frac{\partial \psi_{xj}}{\partial y} + \frac{\partial \psi_{yj}}{\partial x} \right) , \qquad j = 1,2,3$$

$$Q_{xj} = \kappa^2 G_j h \left(\frac{\partial w_j}{\partial x} + \psi_{xj} \right) , \qquad j = 1,2,3$$

Upon substitution of equations (2.48) into (2.49), we obtain a set of simultaneous, linear, algebraic equations

$$[\text{illegible}] \tag{2.50}$$

where for $j = 1,2,3$

$$m = \frac{E}{E_1} = \frac{E_2}{E_3}$$

$$p_{kj} = \frac{C_{kj}}{\Omega} \cdot \frac{n\pi}{a} \qquad k=1,2 \qquad p_{3j} = -\frac{\Omega}{C_{3j}} \frac{a}{n\pi}$$

$$q_{kj} = \frac{C_{kj}}{\Omega} \frac{\xi_{kj}^2}{\xi_{kj}^2 - \rho^2 + \kappa^2 G_j} \qquad k=1,2 \qquad q_{3j} = 0$$

$$\gamma_{kj} = \frac{\Omega}{C_{kj}} + \nu \frac{n\pi}{a} p_{kj} \quad , \qquad k=1,2,3$$

$$f_{kj} = \frac{n\pi}{a} + \frac{\Omega}{C_{kj}} p_{kj} \quad , \qquad k=1,2,3$$

$$g_{kj} = \frac{\Omega}{C_{kj}} q_{kj} + 1 \quad , \qquad k=1,2,3$$

$$S_k^- = e^{\frac{i\Omega\ell}{C_{x2}}} \quad , \quad S_k^- = e^{-\frac{i\Omega}{C_{k2}}} \quad , \quad k=1,2,3 \quad .$$

The quantities A_{11}^-, A_{21}^-, A_{31}^-, A_{12}^+, A_{22}^+, A_{32}^+, A_{12}^-, A_{22}^-, A_{32}^-, A_{13}^+, A_{23}^+, and A_{33}^+ are obtained by the application of Cramer's rule.

The transmission and reflection coefficients in this case are defined as

$$T \equiv \frac{\overline{J_3^+(\ell,t)}}{\overline{J_1^+(0,t)}}$$

$$R \equiv \frac{\overline{J_1^-(0,t)}}{\overline{J_1^+(0,t)}} \quad ,$$

where $J_3^+(x,t)$ is defined in a manner similar to equation (2.19). Numerical results are shown in Figures 2.8 and 2.9.

In elementary theory, the displacement equations, in this case, are

$$\left.\begin{aligned} w_1^+(x,y,t) &= \sum_{k=1}^{2} R_{k1}^+ \, e^{i\Omega\left(t - \frac{x}{C_{k1}}\right)} \sin\frac{n\pi}{a} y \\ w_1^-(x,y,t) &= \sum_{k=1}^{2} R_{k1} \, e^{i\Omega\left(t + \frac{x}{C_{k1}}\right)} \sin\frac{n\pi}{a} y \end{aligned}\right\} , \quad -\infty < x < 0$$

$$\left.\begin{aligned} w_2^+(x,y,t) &= \sum_{k=1}^{2} R_{k2}^+ \, e^{i\Omega\left(t - \frac{x}{C_{k2}}\right)} \sin\frac{n\pi}{a} y \\ w_2^-(x,y,t) &= \sum_{k=1}^{2} R_{k2}^- \, e^{i\Omega\left(t + \frac{x}{C_{k2}}\right)} \sin\frac{n\pi}{a} y \end{aligned}\right\} , \quad 0 < x < \ell \tag{2.52}$$

$$w_3^+(x,y,t) = \sum_{k=1}^{2} R_{k3}^+ \, e^{i\Omega\left(t - \frac{x-}{C_{k3}}\right)} \sin\frac{n\pi}{a} y \quad , \quad \ell < x < \infty$$

where

$$C_{1j}^2 = \frac{\Omega^2}{\Omega\sqrt{\frac{\rho_j h}{D_j}} - \left(\frac{n\pi}{a}\right)^2} \quad , \quad C_{2j}^2 = \frac{\Omega^2}{-\Omega\sqrt{\frac{\rho_j h}{D_j}} - \left(\frac{n\pi}{a}\right)^2} \ ,$$

$$j = 1,2,3$$

The corresponding boundary conditions are

$$\begin{aligned} w_1(0,y,t) &= w_2(0,y,t) \\ \frac{\partial w_1(0,y,t)}{\partial x} &= \frac{\partial w_2(0,y,t)}{\partial x} \\ M_{x1}(0,y,t) &= M_{x2}(0,y,t) \\ \left[Q_{x1} + \frac{\partial M_{xy1}}{\partial y}\right]_{x=0} &= \left[Q_{x2} + \frac{\partial M_{xy2}}{\partial y}\right]_{x=0} \end{aligned} \tag{2.53}$$

$$
\begin{aligned}
w_2(\ell,y,t) &= w_3(\ell,y,t) \\
\frac{\partial w_2(\ell,y,t)}{\partial x} &= \frac{\partial w_3(\ell,y,t)}{\partial x} \\
M_{x2}(\ell,y,t) &= M_{x3}(\ell,y,t) \\
\left[Q_{x2} + \frac{\partial M_{xy2}}{\partial y}\right]_{x=\ell} &= \left[Q_{x3} + \frac{\partial M_{xy3}}{\partial y}\right]_{x=\ell} ,
\end{aligned}
$$

and we obtain the following simultaneous, linear, algebraic equations:

$$
\begin{bmatrix}
[-1] & [1] & [1] & [0] \\
\left[\frac{1}{C_{k1}}\right] & \left[\frac{1}{C_{k2}}\right] & \left[-\frac{1}{C_{k2}}\right] & [0] \\
[-\rho_{k1}] & [m\rho_{k2}] & [m\rho_{k2}] & [0] \\
[\eta_{k1}] & [m\eta_{k2}] & [-m\eta_{k2}] & [0] \\
[0] & [S_k^-] & [S_k^+] & [-1] \\
[0] & \left[\frac{1}{C_{k2}} S_k^-\right] & \left[-\frac{1}{C_{k2}} S_k^+\right] & \left[-\frac{1}{C_{k3}}\right] \\
[0] & [m\rho_{k2} S_k^-] & [m\rho_{k2} S_k^+] & [-\rho_{k3}] \\
[0] & [m\eta_{k2} S_k^-] & [-m\eta_{k2} S_k^+] & [-\eta_{k3}]
\end{bmatrix}
\begin{bmatrix}
R_{11}^- \\ R_{21}^- \\ R_{12}^+ \\ R_{22}^+ \\ R_{12}^- \\ R_{22}^- \\ R_{13}^+ \\ R_{23}^+
\end{bmatrix}
=
\begin{bmatrix}
R_{11}^+ \\ \frac{1}{C_{11}} R_{11}^+ \\ \rho_{11} R_{11}^+ \\ \eta_{11} R_{11}^+ \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} . \tag{2.54}
$$

Numerical values of transmission coefficients resulting from the elementary theory are also shown in Figures 2.8 and 2.9 for comparison purposes.

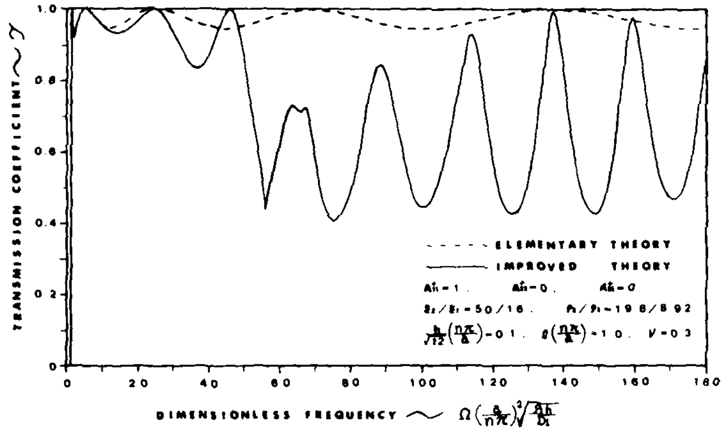

Figure 2.8 - Transmission Coefficient vs. Frequency, Copper-Tungsten-Copper Bonded Plates

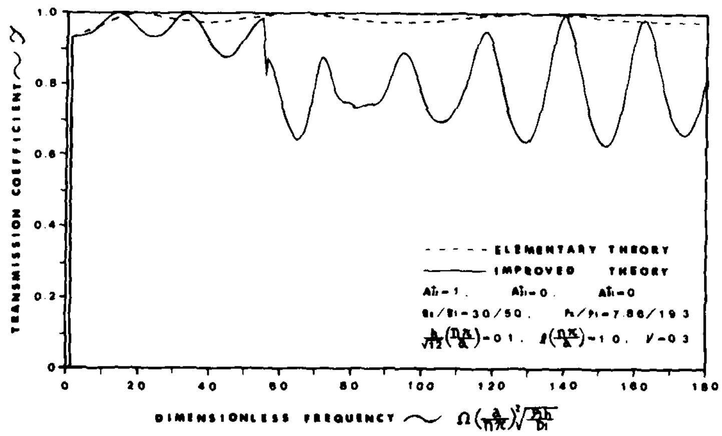

Figure 2.9 - Transmission Coefficient vs. Frequency, Tungsten-Iron-Tungsten Bonded Plates

7. CONCLUSIONS AND DISCUSSION OF RESULTS (PART II)

7.1 *Homogeneous Plates of Unbounded Length*

In the case of elementary plate theory there will be a traveling wave as well as an attenuated standing wave. In response to harmonic excitation, there will be a single traveling sinusoid if

$$\Omega > \Omega_{1C} = \sqrt{\frac{D}{\rho h}}\left(\frac{n\pi}{2}\right)^2 \quad ,$$

where Ω_{1C} is the cut-off frequency. For the case $0 < \Omega < \Omega_{1C}$ there are no traveling waves and both "waves" will be standing and attenuated.

A study of improved plate theory indicates the existence of three different cut-off frequencies $0 < \Omega_{1C} < \Omega_{3C} < \Omega_{2C}$, where Ω_{1C}, Ω_{3C}, Ω_{2C} satisfy the functions

$$\xi_k(\Omega_{kC}) = \frac{n\pi}{a} \quad , \quad (k = 1,2,3)$$

which we can obtain from equation (2.34). In the frequency range $0 < \Omega < \Omega_{1C}$ there are no traveling waves, and we have three standing, attenuated waves. When $\Omega_{1C} < \Omega < \Omega_{3C}$ there exists a single traveling wave and two standing attenuated waves. For $\Omega_{3C} < \Omega < \Omega_{2C}$, we have two traveling waves and a single standing attenuated wave. When $\Omega_{2C} < \Omega$ the three resulting waves are of the traveling variety. Plots of the phase velocity for traveling waves are shown in Figure 2.1. A dimensionless plot of the three cut-off frequencies vs. plate thickness is shown in Figure 2.10. The cut-off frequencies Ω_{2C} and Ω_{3C} become unbounded as $h \to 0$. However,

$$\Omega_{1C} \to \sqrt{\frac{D}{\rho h}}\left(\frac{n\pi}{a}\right)^2$$

as $h \to 0$. These will correspond to the case of classical plate theory.

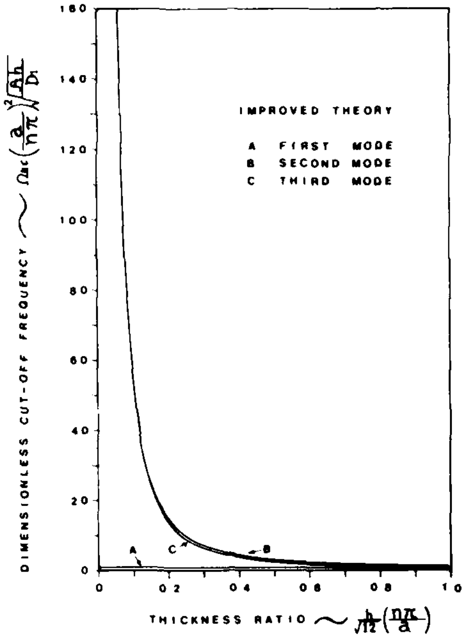

Figure 2.10 - Cut-Off Frequency vs. Thickness

7.2 *Two Bonded, Semi-Infinite Plates Composed of Different Materials*

The incoming waves are assumed to be sinusoidal. With reference to Figures 2.3 through 2.6 and to Figure 2.10, we note that transmission coefficients T of improved plate theory approach those of elementary plate theory as the thickness of the plate approaches zero. We also note that the two theories are in agreement when

$$\Omega < \Omega_{1C} = \sqrt{\frac{D}{\rho h}}\left(\frac{n\pi}{a}\right)^2$$

In the high frequency range $\Omega_{2C} < \Omega$, T for improved theory approaches a constant value which is lower than the T for elementary theory.

In the case of improved theory, the transmission coefficients for each of the three modes approach the same constant value for sufficiently high frequencies, and this value is independent of the plate thickness.

An incoming wave is not possible for $0 < \Omega < \Omega_{1C}$. In the frequency range $\Omega_{1C} < \Omega < \Omega_{3C}$ both theories predict a single, traveling wave, and the transmission coefficients T as calculated for both theories are in good agreement. Differences for T as calculated by the two theories becomes appreciable for $\Omega_{3C} < \Omega$.

7.3 *Plate of Finite Length Bonded to Two Semi-Infinite Plates Composed of a Different Material*

With reference to Figures 2.8 and 2.9, we note that there is good to fair agreement between the two theories in the low frequency range $\Omega_{1C} < \Omega < \Omega_{3C}$. In the high frequency range there can be considerable differences between two theories.

The numbers of maxima (and minima) of T per unit frequency region in improved theory approaches a constant value as the frequencies increase, while in elementary theory they decrease as the frequencies increase. This phenomenon is a consequence of the difference in dispersion relations of the two theories.

ACKNOWLEDGMENT

This project was supported by the United States Air Force Office of Scientific Research under Grant No. AF-AFOSR-76-2943.

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NOMENCLATURE - PART II

Quantity	*Description*
a	width of a plate
C_{kj}	phase velocity
D	flexural rigidity of plate ($D=Eh^3/12(1-\nu^2)$)
E	Young's Modulus
e	exponential function

e_{xy}	components of strain tensor
G	modulus of elasticity in shear $(G=E/2(1+\nu))$
h	thickness of a plate
i	$\sqrt{-1}$
J	energy flux of a plate
j	domain of a plate
k	mode of motion
ℓ	length of a finite plate strip
M_x, M_y, M_{xy}	bending and twisting moments of a plate
M_x^*, M_y^*, M_{xy}^*	intensities of bending and twisting moment at boundary
m	ratio of Young's Modulus $(m = E_2/E_1)$
p	intensity of applied surface load
Q_x, Q_y	shearing forces of a plate
Q_x^*, Q_y^*	intensities of shearing force at boundary
R	*reflection coefficient*
S_x, S_y	intensities of applied shearing force on surface
T	kinetic energy
T	transmission coefficient
t	time
U_x, U_y, U_z	displacements in x,y and z direction
V	potential energy
δW	virtual work
w	warping displacement of plate median surface
x, y, z	cartesian coordinates
κ^2	Mindlin's correction factor

ν	Poisson's ratio
ρ	density
σ_{xy}	components of stress tensor
ψ_x, ψ_y	rotation angles of line originally normal to median surface
ω	frequency
ω_{kc}	the kth mode cut-off frequency

Reprint from

SM Archives

Noordhoff International Publishing
Leyden, The Netherlands

SM Archives, Vol. 2, Issue 2, May 1977
Noordhoff International Publishing - Alphen aan den Rijn
Printed in the Netherlands

DYNAMICS OF INITIALLY STRESSED HYPERELASTIC SOLIDS

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1. INTRODUCTION

The two principal objectives of the present report are: (1) to logically develop a linear theory for the dynamics of initially stressed elastic solids, and (2) to present a variational principle which will serve as the framework for the systematic development of approximate theories for the incremental motion of prestressed rods, beams, plates, shells, etc. Additionally, the derivations will clearly show how the apparent mechanical properties of the solid are altered by the pre-stress. To accomplish these objectives, the following assumptions will be made:

(1) The solid is hyperelastic, i.e., it possesses a natural state in which all stresses and strains vanish simultaneously, and its mechanical constitution is completely characterized by a strain energy density function which vanishes in the natural state.

(2) An arbitrary pre-stress causes the initial deformation of the solid from its natural state to the

initial state. This initial configuration corresponds to a static equilibrium state of the solid which is not necessarily stable.

(5) The incremental motion of the solid about its initial configuration shall be sufficiently small to warrant linearization of the resulting theory on the basis of small incremental deformations. Note: The magnitude of the initial deformation is arbitrary, only the incremental deformation is assumed to be *small*.

The approach to be followed in the derivation of the basic equations is as follows. The total deformation of the solid is separated into two parts: (1) the initial deformation, and (2) the incremental deformation. The strain energy density function and the constitutive relation are both developed as power series in the incremental deformation about the initial state. Based on the assumption of small incremental deformations, the power series for the strain energy density is terminated after the quadratic terms and the power series for the constitutive relation is terminated after the linear terms. This results in a linear relation between the incremental stress and the incremental deformation. It also clearly shows how the mechanical properties relating the incremental stress and deformation differ from those measured in the natural state. For example, a solid which is isotropic in its natural state may appear to be anisotropic when studied in its initial state. Furthermore, quantities such as Young's modulus, Poisson's ratio, dilational and rotational wave speeds, etc., when measured from the initial state, will differ from the value obtained by measurements made with respect to the *natural* state. The truncated power series for the strain energy density function is next substituted into Hamilton's Principle. This yields the linear equations of motion and proper natural boundary conditions for the incremental motion. At this point a variational principle governing the linear theory of pre-stressed elastic solids is presented. This result provides the logical framework for the systematic development of special approximate theories for pre-stressed rods, beams, plates, shells, etc. A resolution of the resulting boundary value problem for the incremental motion of solids of bounded extent is obtained by the application of classical, mathematical techniques. In order to illustrate how the results of this investigation can be applied, two specific examples are presented: (a) motion of a solid subjected to an initial hydrostatic pressure, and (b) the motion of an initially stressed beam.

Problems relating to the behavior of pre-stressed elastic solids are of considerable interest to both engineers and physicists. Engineers are routinely faced with the problem of analyzing and designing structures containing pre-stressed members. This invariably leads to the question of stability (or instability) of the initial configuration. The proper way to assess stability is to subject the pre-stressed solid to incremental surface tractions and/or initial conditions and to observe whether or not the resulting incremental motion remains close to the initial configuration. If it does then the initial configuration is stable in the sense of Lagrange and Liapounov. The closeness referred to above must be defined analytically in terms of a specific measure of the incremental motion, e.g., the squared norm of the incremental displacement. Moreover, if the external forces which induce the pre-stress are conservative, then the initial state is also stable in the classical Euler-Lagrange sense, i.e., the potential energy of the system assumes at the equilibrium position a weak relative minimum in the class of virtual displacements satisfying the kinematical constraints. An excellent treatise on the theory of elastic stability by Knops [1] appears in S. Flügge's Encyclopedia of Physics. He investigates both the Liapounov criteria and the energy criteria for stability and shows when the two are equivalent and when they are not. Bolotin [2] has also investigated stability from the dynamical viewpoint with emphasis on the study of non-conservative external forces. In addition to questions of stability, the study of pre-stressed solids has been pursued because of its applications to the field of crystal physics. The material characteristics of solids subject to initial hydrostatic pressure have been used to test various atomic model potentials for crystals, to study anharmonic crystals, and to study various thermal and electrical properties of crystals. The field of acoustics has also come to play an important role in the present subject because most of the experimental techniques used to study the material behavior of pre-stressed solids are based on ultrasonic methods.

In view of the importance of the applications mentioned above, a great deal of literature on the subject has developed. Here we will cite only those references most directly relevant to the present treatment. A more extensive list of references may be found in [1] and also in the treatise on non-linear field theories of mechanics by Truesdell and Noll [3]. As pointed out in [3], Cauchy successfully derived a linear theory of incremental motion superimposed on a pre-stressed elastic body. Cauchy's equations are given by Todhunter and Pearson [4]. Saint-Venant [4] attempted to duplicate Cauchy's results using an energy principle. However, his results were not valid because he did not retain the quadratic terms in his expansion of the strain energy. More recent derivations of the linear theory for incremental motion superimposed on finitely deformed elastic bodies were given by Toupin and Bernstein [5], Truesdell and Noll [3], Green, Rivlin and Shield [28], Green and Zerna [29], Knops [1], Eringen and Suhubi [6]. Of these, the one which is probably closest in spirit to the present variational approach is the derivation given by Knops based on the invariance of the rate of work equations under arbitrary rigid body motions. The results of [28] and [29] are identical to the ones in Section 2 of the present investigation, but they are obtained in an entirely different manner. Applications of these results to particular cases of initially stressed solids can be found in [30] through [35]. We also mention the contributions of Biot which are summarized in [36]. In this latter treatment of the initially stressed solid, variables are tailored to the specific asymmetry of the physics and geometry of the problems at hand. In many cases, such variables are not tensors. By sacrificing the requirements of invariance, this work does not seem to fit into the main stream of modern continuum mechanics.

2. ANALYSIS AND VARIATIONAL FORMULATION

Throughout the forthcoming analysis, standard Cartesian tensor notation will be used, with all particle coordinates and tensor components referred to the same, fixed, rectangular Cartesian coordinate system. Three different configurations of the hyperelastic, solid body must be distinguished:

(1) In its natural state, the body occupies the region V_a bounded by the surface S_a and the coordinates of a typical material particle P are denoted by a_i. The stress, strain, and strain energy density all vanish identically throughout V_a.
(2) In its initial (static equilibrium) state, the body occupies the region V_o bounded by the surface S_o and the coordinates of P are denoted by x_i^o. The Cauchy stress tensor, Lagrangian strain tensor, and strain energy density are denoted, respectively, by τ_{ij}^o, E_{ij}^o, and W_o^*.
(3) In the current (or final) state, the body occupies the region V_F bounded by the surface S_F and the coordinates of P are denoted by x_i^F. The Cauchy stress tensor, Lagrangian strain tensor, and strain energy density are denoted, respectively, by τ_{ij}^F, E_{ij}^F, and W_F^*.

The initial deformation carries the material particle P from its location P_a (a_1, a_2, a_3) in the natural configuration to the location P_o (x^o_1, x^o_2, x^o_3) in the initial configuration. The incremental deformation then carries it to its final location P_F (x^F_1, x^F_2, x^F_3). The components of the incremental displacement vector $\vec{P_oP_F}$ will be denoted by u_i, i.e.,

$$u_i = x^F_i - x^o_i \tag{2.1}$$

With respect to the natural configuration, the initial and current configurations are described by the one-to-one mappings

$$x^o_i = x^o_i\,(a_1, a_2, a_3) \tag{2.2a}$$

$$x^F_i = x^F_i\,(a_1, a_2, a_3, t) \tag{2.2b}$$

Conversely, with respect to the initial configuration,

$$a_i = a_i\,(x^o_1, x^o_2, x^o_3) \tag{2.3}$$

and therefore (2.2b) can also be written as

$$x^F_i = x^F_i\,(x^o_1, x^o_2, x^o_3, t) \tag{2.4}$$

Since the results of any measurements performed on a pre-stressed body are obtained in terms of the initial coordinates, we shall ultimately express all results in terms of these coordinates using (2.3) and (2.4).

The Lagrangian strain tensor, Cauchy stress tensor, and constitutive relations are defined by the familiar relations (see for example Chapter 9 of [7]):

$$E^F_{ij} = \frac{1}{2}\left(\frac{\partial x^F_k}{\partial a_i}\frac{\partial x^F_k}{\partial a_j} - \delta_{ij}\right) \tag{2.5}$$

$$t^F_{ij} = J_F^{-1}\,\frac{\partial x^F_i}{\partial a_k}\frac{\partial x^F_j}{\partial a_\ell}\,T^F_{k\ell} \tag{2.6}$$

$$T^F_{k\ell} = \left(\frac{\partial W^*}{\partial E_{k\ell}}\right)_F \tag{2.7}$$

In (2.7), $T_{k\ell}$ is the symmetric, Piola-Kirchhoff stress tensor and

[illegible]

$$W^* = W^*(E_{ij}, a_i) \tag{2.8}$$

is the strain energy density function, i.e., the strain energy per unit volume of the natural state. In (2.6), $J_F^{-1} = \frac{1}{J_F}$ where $J_F = \det(\partial x_i^F / \partial a_j)$ is the *Jacobian determinant* of the transformation (2.2b). Upon substitution of (2.1) into (2.5) we find that the total strain can be written as the sum of the initial strain plus an incremental strain,

$$\text{i.e.,} \qquad E_{ij}^I = E_{ij}^o + \frac{\partial x_k^o}{\partial a_i} \frac{\partial x_l^o}{\partial a_j} E_{kl} \tag{2.9a}$$

$$\text{where,} \qquad E_{ij}^o = \frac{1}{2}\left(\frac{\partial x_k^o}{\partial a_i} \frac{\partial x_k^o}{\partial a_j} - \delta_{ij}\right) \tag{2.9b}$$

is the initial strain and,

$$E_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k} + u_{i,k} u_{i,l}) \tag{2.9c}$$

is the incremental strain referred to the initial state (a comma is used to denote partial differentiation with respect to the initial coordinates x_j^o, i.e., $\partial u_k / \partial x_l^o = u_{k,l}$).

We now assume that the strain energy density function together with its partial derivatives up to and including the third order in strain, exist and are continuous at E_{ij}^o (the initial state). In this case we can use Taylor's theorem to approximate W^*, in the neighborhood of the initial state, by a function which is quadratic in the incremental strain, i.e.,

$$W^*(E_{ij}^F) \approx W^*(E_{ij}^o) + \left(\frac{\partial W^*}{\partial E_{ij}}\right)_o (E_{ij}^I - E_{ij}^o)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 W^*}{\partial E_{ij} \partial E_{kl}}\right)_o (E_{ij}^I - E_{ij}^o)(E_{kl}^I - E_{kl}^o)$$

or in view of (2.9a),

$$W_F^* = W_o^* + T_{ij}^o \frac{\partial x_k^o}{\partial a_i} \frac{\partial x_\ell^o}{\partial a_j} E_{k\ell} + \frac{1}{2} A_{ijk\ell}^o \frac{\partial x_p^o}{\partial a_i} \frac{\partial x_q^o}{\partial a_j} \frac{\partial x_r^o}{\partial a_k} \frac{\partial x_s^o}{\partial a_\ell} E_{pq} E_{rs}$$

$$= W_o^* + J_o \left(\tau_{ij}^o E_{ij} + \frac{1}{2} B_{ijk\ell}^o E_{ij} E_{k\ell}\right) \tag{2.10}$$

where

$$T_{ij}^o = \left(\frac{\partial W^*}{\partial E_{ij}}\right)_o \quad \text{is the symmetric Piola-Kirchoff pre-stress tensor,} \tag{2.11a}$$

$$\tau_{ij}^o = J_o^{-1} \frac{\partial x_i^o}{\partial a_k} \frac{\partial x_j^o}{\partial a_\ell} T_{k\ell}^o \quad \text{is the Cauchy pre-stress tensor,} \tag{2.11b}$$

$$A_{ijk\ell}^o = \left(\frac{\partial^2 W^*}{\partial E_{ij} \partial E_{k\ell}}\right)_o = A_{k\ell ij}^o = A_{jik\ell}^o = A_{ij\ell k}^o \tag{2.11c}$$

$$B_{ijk\ell}^o = J_o^{-1} \frac{\partial x_i^o}{\partial a_p} \frac{\partial x_j^o}{\partial a_q} \frac{\partial x_k^o}{\partial a_r} \frac{\partial x_\ell^o}{\partial a_s} A_{pqrs}^o \tag{2.11d}$$

and

$$J_o = \det\left(\frac{\partial x_i^o}{\partial a_j}\right) . \tag{2.11e}$$

Next, the strain-displacement relation (2.9c) is substituted into (2.10) and the assumption of small incremental deformations and rotations is imposed to obtain, after neglecting third and fourth order products of the derivatives $u_{i,j}$:

$$W_F^* = W_o^* + J_o \tau_{ij}^o e_{ij} + \frac{1}{2} J_o \left[\tau_{ij}^o u_{k,i} u_{k,j} + B_{ijk\ell}^o e_{ij} e_{k\ell}\right] \tag{2.12}$$

where

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) . \tag{2.13}$$

In view of this result, the total strain energy of the body is given by

$$U_F = \int_{V_a} W_F^* dV_a = U_o^* + \int_{V_o} \tau_{ij}^o e_{ij} dV_o + U \tag{2.14a}$$

where

$$U_o^* = \int_{V_a} W_o^* dV_a \tag{2.14b}$$

is the strain energy associated with the initial deformation, and

$$U = \frac{1}{2} \int_{V_o} (\tau_{ij}^o u_{k,i} u_{k,j} + B_{ijk\ell}^o e_{ij} e_{k\ell}) dV_o \tag{2.14c}$$

is the incremental strain energy. To obtain these results we made use of the relation $dV_o = J_o dV_a$ to transform the volume integrals from the natural to the initial configuration.

Consistent with our quadratic approximation for W_F^*, we can also write

$$T_{ij}^F = \left(\frac{\partial W^*}{\partial E_{ij}}\right)_F = \left(\frac{\partial W^*}{\partial E_{ij}}\right)_o + \left(\frac{\partial^2 W^*}{\partial E_{ij} \partial E_{k\ell}}\right)_o (E_{k\ell}^F - E_{k\ell}^o)$$

$$= T_{ij}^o + A_{ijk\ell}^o \frac{\partial x_r^o}{\partial a_k} \frac{\partial x_s^o}{\partial a_\ell} E_{rs}$$

$$= J_o \frac{\partial a_i}{\partial x_p^o} \frac{\partial a_j}{\partial x_q^o} (\tau_{pq}^o + B_{pqrs}^o E_{rs}).$$

Therefore,

$$\tau_{ij}^F = J_F^{-1} \frac{\partial x_i^F}{\partial a_k} \frac{\partial x_j^F}{\partial a_\ell} T_{k\ell}^F$$

$$= J^{-1} x_{i,p}^F x_{j,q}^F (\tau_{pq}^o + B_{pqrs}^o E_{rs}) \tag{2.15}$$

In view of (2.1), $x_{i,p}^F = \delta_{ip} + u_{i,p}$, and thus for small incremental deformations and rotations,

$$x_{i,p}^F x_{j,q}^F = \delta_{ip} \delta_{jq} + \delta_{ip} u_{j,q} + \delta_{jq} u_{i,p} . \tag{2.16}$$

In (2.15), $J^{-1} = J_F^{-1} J_o$ and therefore $J = J_F J_o^{-1} = \det(x^F_{i,j}) = \det(\delta_{ij} + u_{i,j})$ is the Jacobian of the incremental deformation. For small incremental deformations and rotations,

$$J = 1 + u_{k,k} = 1+e \qquad \text{and therefore}$$

$$J^{-1} = 1 - u_{k,k} = 1-e \,. \tag{2.17}$$

Substitution of (2.16) and (2.17) into (2.15) yields, after neglecting second and higher order products of the derivatives $u_{i,j}$,

$$\tau^F_{ij} = \tau^o_{ij} + \tau_{ij} \tag{2.18a}$$

where

$$\tau_{ij} = \tau^o_{ik} u_{j,k} + \tau^o_{kj} u_{i,k} - \tau^o_{ij} u_{k,k} + B^o_{ijk\ell} e_{k\ell} \tag{2.18b}$$

or, in more compact form,

$$\tau_{ij} = C^o_{ijk\ell} u_{k,\ell} \tag{2.18c}$$

with

$$C^o_{ijk\ell} = \tau^o_{i\ell} \delta_{jk} + \tau^o_{\ell j} \delta_{ik} - \tau^o_{ij} \delta_{k\ell} + B^o_{ijk\ell}$$

$$= C^o_{jik\ell} \neq C^o_{ij\ell k} \,. \tag{2.18d}$$

Equation (2.18b or c) is *the constitutive relation* for a small incremental deformation superimposed upon a finite initial deformation. It was first derived by Cauchy in 1828 [1]. However, Cauchy's derivation of (2.18b) was more general than ours in that he did *not* appeal to the theory of elasticity but rather showed that (2.18b) is the most *general bilinear function* of $u_{i,j}$ and τ^o_{ij} that satisfies the principle of material frame indifference. See the note on page 250 of [3] in this regard.

The constitutive relation (2.18c) can be rewritten in terms of the incremental strains and rotations according to the following scheme suggested by Drabble [8]:

$$\sigma_{ij} = S^{o}_{ijk\ell} e_{k\ell} + R^{o}_{ijk\ell} \omega_{k\ell} \tag{2.19a}$$

where

$$\omega_{k\ell} = u_{k,\ell} - e_{k\ell} = \frac{1}{2}(u_{k,\ell} - u_{\ell,k}) \tag{2.19b}$$

is the skew-symmetric, incremental rotation tensor, and

$$S^{o}_{ijk\ell} = B^{o}_{ijk\ell} + \frac{1}{2}(\sigma^{o}_{ik}\delta_{j\ell} + \sigma^{o}_{i\ell}\delta_{jk} + \sigma^{o}_{kj}\delta_{i\ell} + \sigma^{o}_{\ell j}\delta_{ik} - 2\sigma^{o}_{ij}\delta_{k\ell}) \tag{2.19c}$$

$$R^{o}_{ijk\ell} = \frac{1}{2}(\sigma^{o}_{i\ell}\delta_{jk} - \sigma^{o}_{ik}\delta_{j\ell} + \sigma^{o}_{\ell j}\delta_{ik} - \sigma^{o}_{kj}\delta_{i\ell}) \; . \tag{2.19d}$$

These coefficients possess the symmetr properties

$$\begin{aligned} S^{o}_{ijk\ell} &= S^{o}_{jik\ell} = S^{o}_{ij\ell k} \\ R^{o}_{ijk\ell} &= R^{o}_{jik\ell} = -R^{o}_{ij\ell k} \; . \end{aligned} \tag{2.19e}$$

It is evident from (2.19) that the incremental stress cannot be related solely to the incremental strain unless the pre-stress tensor happens to be spherical (hydrostatic pressure or tension) or else vanishes. Equations (2.11) and (2.19) reveal precisely how the pre-stress alters the apparent mechanical behavior of the solid. For example, a material which is isotropic in its natural state will appear to be anisotropic with regard to incremental deformations observed from the initial state unless the pre-stress tensor happens to be spherical, and even in that case, the magnitudes of the elastic constants will be altered in accordance with (2.11).

We shall now proceed to derive the linear equations of incremental motion for the initially stressed solid using Hamilton's Principle. As shown in [7], Hamilton's Principle for a hyperelastic solid takes the form

$$\int_{t_1}^{t_2} \delta L_F dt = - \int_{t_1}^{t_2} \delta W_F dt \quad , \tag{2.20}$$

where $L_F = T_F - U_F$ is the Lagrangian function corresponding to the current state of the body. Similarly, T_F and U_F are the total kinetic energy and total strain energy associated with the current state of the body. The variation $\delta L_F = \delta T_F - \delta U_F$ results from allowing small variations of the incremental displacement vector. These variations, δu_i, are arbitrary throughout V_o, consistent with the boundary conditions on S_o, and vanish at the instants of time t_1 and t_2. The quantity δW_F is the work done by the external forces as a result of the variation δu_i. (These forces need not be conservative). In view of (2.14),

$$\delta U_F = \delta U_o + \delta U \tag{2.21}$$

where

$$\delta U_o = \int_{V_o} \tau^o_{ij} (\delta u_i)_{,j} dV_o$$

$$= \int_{S_o} T^o_i \delta u_i ds_o + \int_{V_o} \rho_o f^o_i \delta u_i dV_o \tag{2.22a}$$

and

$$\delta U = \int_{V_o} D^o_{ijk\ell} u_{k,\ell} (\delta u_i)_{,j} dV_o$$

$$= \int_{S_o} D_{ijk\ell} u_{k,\ell} \delta u_i n^o_j dS_o - \int_{V_o} (D^o_{ijk\ell} u_{k,\ell})_{,j} \delta u_i dV_o \tag{2.22b}$$

where

$$D^o_{ijk\ell} = \tau^o_{j\ell}\delta_{ik} + B_{ijk\ell} = D^o_{k\ell ij} \qquad (2.23a)$$

$$= C^o_{ijk\ell} - \tau^o_{i\ell}\delta_{jk} + \tau^o_{ij}\delta_{k\ell} \qquad (2.23b)$$

In the derivation of the second equation in (2.22a) and (2.22b) we made use of Gauss' Theorem to transform from volume to surface integrals and we also imposed the equilibrium conditions on the initial configuration, i.e.,

$$\tau^o_{ij,j} = -\rho_o f^o_i \quad \text{in } V_o \qquad (2.24a)$$

$$\tau^o_{ij} n^o_j = T^o_i \quad \text{on } S_o \quad . \qquad (2.24b)$$

With the aid of (2.19a), (2.23), and (2.24), the variation δU can also be expressed in terms of the incremental stress as follows:

$$\delta U = \int_{S_o} (\tau_{ij} n^o_j - \tau^o_{ik} u_{j,k} n^o_j + T^o_i e)\delta u_i dS_o - \\ - \int_{V_o} (\tau_{ij,j} - \tau^o_{ik,j} u_{j,k} - \rho_o f^o_i e)\delta u_i dV_o \qquad (2.25)$$

where $e = e_{kk} = u_{k,k}$.

The total kinetic energy of the body is

$$T_F = \frac{1}{2}\int_{V_F} \dot{x}^F_i \dot{x}^F_i \rho_F dV_F = \frac{1}{2}\int_{V_o} \dot{u}_i \dot{u}_i \rho_o dV_o$$

and, therefore,

$$\delta T_F = \int_{V_o} \dot{u}_i \delta\dot{u}_i \rho_o dV_o = \delta T \quad . \qquad (2.26)$$

Since $\delta u_i \equiv 0$ at $t = t_1$ and $t = t_2$,

$$\int_{t_1}^{t_2} \delta T dt = - \int_{t_1}^{t_2}\int_{V_o} \ddot{u}_i \delta u_i \rho_o dV_o dt \qquad (2.27)$$

where a dot denotes the partial derivative with respect to time. The external forces currently acting on the body consist of a surface traction T^F_i and a body force f^F_i. The work done by these forces during the virtual displacement δu_i is given by

$$\delta W_F = \int_{S_F} T^F_i \delta u_i dS_F + \int_{V_F} \rho_F f^F_i \delta u_i dV_F \qquad (2.28a)$$

$$= \int_{S_o} {}^oT^F_i \delta u_i dS_o + \int_{V_o} \rho_o f^F_i \delta u_i dV_o \qquad (2.28b)$$

where we have used the conservation of mass $\rho_o dV_o = \rho_F dV_F$ to transform the volume integral from V_F to V_o, and we have also introduced the notation ${}^oT^F_i$ to represent the value of the final contact force per unit initial area, i.e.,

$${}^oT^F_i = T^F_i \frac{dS_F}{dS_o} . \qquad (2.29)$$

The elements of surface dS_F and dS_o are related as follows [7]:

$$n^F_k dS_F = \frac{\partial x^o_i}{\partial x^F_k} J n^o_i dS_o$$

where n^F_k and n^o_i are the unit outward normals to the surfaces S_F and S_o, respectively. Consequently,

$$dS_F = J dS_o \left[\frac{\partial x^o_i}{\partial x^F_k} \frac{\partial x^o_j}{\partial x^F_k} n^o_i n^o_j \right]^{1/2}$$

which, for small incremental deformations, is approximated by

$$dS_F = [1+u_{k,k} - e_{ij} n^o_i n^o_j] dS_o . \qquad (2.30)$$

Substituting (2.30) into (2.29) then yields

$${}^oT^F_i = T^F_i [1+u_{k,k} - e_{ij} n^o_i n^o_j] . \qquad (2.31)$$

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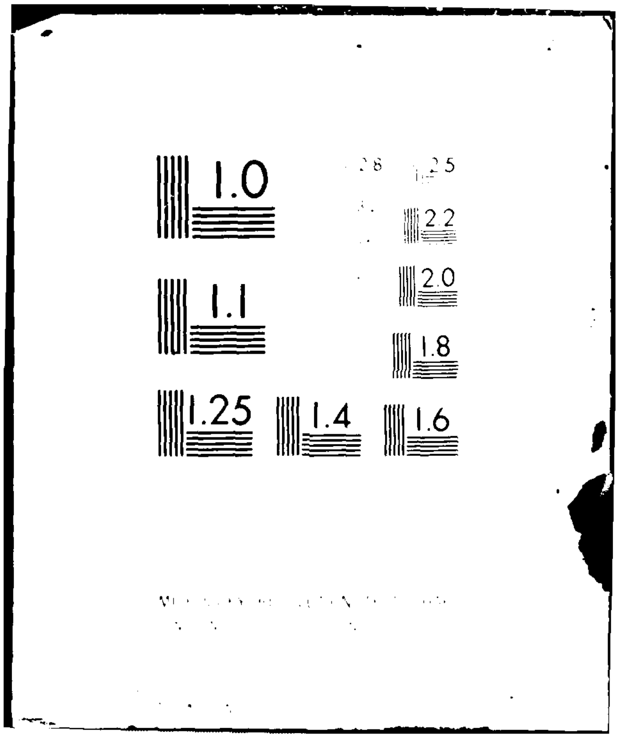
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We now define an incremental surface traction (incremental force per unit initial area) and an incremental body force by the relations

$$
\begin{aligned}
T_i &= T_i^1 - T_i^o \\
f_i &= f_i^1 - f_i^o
\end{aligned}
\tag{2.52}
$$

Substitution of (2.52) into (2.28b) then yields

$$\delta W_1 = \delta W_o + \delta W \tag{2.53}$$

where

$$\delta W_o = \int_{S_o} T_i^o \delta u_i dS_o + \int_{V_o} \rho_o f_i^o \delta u_i dV_o \tag{2.54}$$

and

$$\delta W = \int_{S_o} T_i \delta u_i dS_o + \int_{V_o} \rho_o f_i \delta u_i dV_o \tag{2.55}$$

The incremental surface traction T_i defined by (2.52) corresponds to $\dot{t}_R^{\,i}$ in [5], and to $(t_i^* - t_i)$ in [1]. However, Eringen and Suhubi [6] define the incremental traction $\hat{t}_i$ as simply $T_i^F - T_i^o$. In view of (2.51) and (2.52) we see that our incremental traction is related to theirs through the equation

$$T_i = \hat{t}_i + T_i^o[u_{k,k} - e_{ij}n_i^o n_j^o] \tag{2.56}$$

where we have neglected products of $\hat{t}_i$ with the incremental deformation $u_{i,j}$.

The variational principle governing the incremental motion may be obtained by substituting (2.21), (2.26), and (2.55) into (2.20). With the aid of (2.22a) and (2.54) this results in the variational equation

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)dt = 0. \tag{2.57}$$

The use of (2.22b), (2.27), and (2.55) in (2.57) yields the displacement equations of motion and associated boundary conditions

$$(D^o_{ijk\ell}u_{k,\ell})_{,j} + \rho_o f_i = \rho_o \ddot{u}_i \quad \text{in } V_o \tag{2.58a}$$

$$u_i = F_i(x,t) \quad \text{on } S_{o1} \tag{2.58b}$$

$$D^o_{ijk\ell}u_{k,\ell}n^o_j = T_i \quad \text{on } S_{o2} \tag{2.58c}$$

Similarly, the substitution of (2.25), (2.27), and (2.55) yields the stress equations of motion

$$\tau_{ij,j} - \tau^o_{ik,j}u_{j,k} + \rho_o(f_i - ef^o_i) = \rho_o \ddot{u}_i \quad \text{in } V_o \tag{2.59a}$$

$$u_i = F_i(x,t) \quad \text{on } S_{o1} \tag{2.59b}$$

$$\tau_{ij}n^o_j = T_i + \tau^o_{ik}u^o_{j,k}n^o_j - T^o_i e \quad \text{on } S_{o2} \tag{2.59c}$$

In the preceding equations we assume that S_o is the union of the two disjoint sets S_{o1} and S_{o2}. Note that in the case of a constant initial pre-stress, the stress equation of motion becomes

$$\tau_{ij,j} + \rho_o f_i = \rho_o \ddot{u}_i \tag{2.59d}$$

which is formally the same as in the case of zero pre-stress. However, the boundary condition (2.59c) does not simplify any further. A third form of these equations may be obtained by substituting (2.23a) into (2.58). This yields,

$$(B^o_{ijk\ell}u_{k,\ell})_{,j} + \tau^o_{jk}u_{i,jk} + \rho_o(f_i - f^o_k u_{i,k}) = \rho_o \ddot{u}_i \quad \text{in } V_o \tag{2.40a}$$

$$u_i = F_i(x,t) \quad \text{on } S_{o1} \tag{2.40b}$$

$$B^{o}_{ijk\ell}u_{k,\ell}n^{o}_{j} = T_i - T^{o}_{k}u_{i,k} \qquad \text{on } S_{o2} \tag{2.40c}$$

For many technical applications it can be assumed that

$$|\tau^{o}_{i\ell}\delta_{jk} + \tau^{o}_{\ell j}\delta_{ik} - \tau^{o}_{ij}\delta_{k\ell}| << B^{o}_{ijk\ell}$$

and, with reference to (2.18d), we then have $C^{o}_{ijk\ell} = B^{o}_{ijk\ell}$. As a consequence, using (2.18e), $\tau_{ij} = B^{o}_{ijk\ell}u_{k,\ell}$, and in this case equations (2.40) reduce to

$$\tau_{ij,j} + (\tau^{o}_{jk}u_{i,k})_{,j} + \rho_o f_i = \rho_o \ddot{u}_i$$

$$u_i = F_i(x,t) \qquad \text{on } S_{o1}$$

$$(\tau_{ij} + \tau^{o}_{jk}u_{i,k})n^{o}_{j} = T_i \text{ on } S_{o2}$$

where we have utilized (2.24).

These equations have the same form as (1.34) and (1.35), page 46 of [2]. In the case of a homogeneous body subjected to a constant pre-stress, (2.40) becomes

$$B^{o}_{ijk\ell}u_{k,j\ell} + \tau^{o}_{jk}u_{i,jk} + \rho_o f_i = \rho_o \ddot{u}_i \; . \tag{2.41}$$

This form is identical to equation (iii), page 84 of [4] and is attributed to Cauchy (1829). The system of equations and boundary conditions (2.38) also appear in: [1] (p. 180, (23.17), (23.18), (23.19)); [5] (p. 247, (68.9)); and [6] (p. 253, (4.2.30), also see p. 254, (4.2.34) compared to our (2.40)). The present derivation differs from these others in that it is based on a variational principle (2.37) which, with suitable restrictions on the class of allowable displacements, can be used to derive special approximate theories for pre-stressed rods, beams, plates, and shells.

As a result of the symmetry relation $D^o_{ijkl} = D^o_{klij}$, one can easily show that Betti's reciprocal theorem holds for the incremental motion, i.e.,

$$\int_{V_0} \rho_0 \left(f_i^{(1)} - \ddot{u}_i^{(1)}\right) u_i^{(2)} dV_0 + \int_{S_0} t_i^{(1)} u_i^{(2)}$$

$$= \int_{V_0} \rho_0 \left(f_i^{(2)} - \ddot{u}_i^{(2)}\right) u_i^{(1)} dV_0 + \int_{S_0} t_i^{(2)} u_i^{(1)} \qquad (2.42)$$

Truesdell and Noll [5] show further that, "in order that Betti's theorem shall hold for infinitesimal deformations superimposed on any given configuration of an elastic material, it is necessary and sufficient that the elastic material be hyperelastic." They then use (2.42) to prove that, "for infinitesimal free harmonic vibration about any configuration of a hyperelastic body, the normal functions corresponding to distinct proper frequencies are orthogonal," i.e., when

$$f_i^{(1)} = f_i^{(2)} = t_i^{(1)} = t_i^{(2)} = 0 \text{ and } u_i^{(1)} = U_i^{(1)}(x)\cos \omega_1 t ,$$

$$u_i^{(2)} = U_i^{(2)}(x)\cos \omega_2 t$$

then

$$(\omega_1^2 - \omega_2^2) \int_{V_0} \rho_0 U_i^{(1)} U_i^{(2)} dV_0 = 0$$

or,

$$\int_{V_0} \rho_0 U_i^{(1)} U_i^{(2)} dV_0 = 0 \quad \text{if} \quad \omega_1^2 \neq \omega_2^2 . \qquad (2.43)$$

Truesdell and Noll [5], and Hill [37] derive the following energy criterion for what they call infinitesimal stability of the initial state,

$$2W = \int_{V_0} D^o_{ijkl} u_{i,j} u_{k,l} dV_0 > 0 . \qquad (2.44)$$

where U is the incremental strain energy associated with any incremental deformation which is compatible with the boundary conditions. Knops [1] also obtains (2.44) by requiring that the potential energy be a weak relative minimum at the initial equilibrium state. He refers to this as Hadamard infinitesimal stability since (2.44) necessarily implies that Hadamard's inequality

$$D^o_{ijk\ell}\xi_i\xi_k\eta_j\eta_\ell \geq 0 \tag{2.45}$$

holds at every point of V_o for all vectors ξ_i and η_i. Knops further shows that for conservative loads, (2.44) is necessary but not sufficient to guarantee Liapounov stability. For example, in the case where the initial equilibrium state satisfies displacement or mixed boundary conditions, the inequality

$$\int_{V_o} D^o_{ijk\ell}\xi_{ij}\xi_{k\ell}dV_o \geq d\int_{V_o} \xi_{ij}\xi_{ij}dV_o \tag{2.46}$$

for all symmetric tensors ξ_{ij} and some positive constant d is sufficient to guarantee Liapounov stability. If the initial state is maintained solely by traction boundary conditions, then in addition to (2.46), the restriction

$$\int_{V_o} \dot{u}_i dV = 0 \tag{2.47}$$

is necessary to prove Liapounov stability. The measure chosen in these cases is

$$\rho(u) = \int_{V_o} \rho_o u_i u_i dV_o \tag{2.48}$$

If the criterion for infinitesimal stability (2.44) is satisfied then one can easily show that (2.38) possesses at most one solution if the initial displacement and velocity are specified throughout V_o [3]. Furthermore, the natural frequencies associated with free harmonic incremental motions ($f_i=0$, $T_i=0$, $F_i=0$) of the form $u_i=U_i(x)\cos\omega t$ are real. To prove this, multiply (2.38a) by U_i, integrate over V_o and impose the homogeneous boundary data to obtain

$$\omega^2 \int_{V_o} \rho_o U_i U_i dV_o = 2U \geq 0 \ . \qquad (2.49)$$

Thus, $\omega^2 \geq 0$. In the case of strict inequality, $U>0$, (referred to as superstability in [3]), we further conclude that no natural frequency can vanish.

An energy continuity equation can be derived for the incremental motion by imposing the conservation of energy law in the following form:

$$\frac{dE^*}{dt} = \int_{S_F} T_i^F \dot{u}_i dS_F + \int_{V_F} f_i^F \dot{u}_i dS_F \qquad (2.50)$$

where

$$E^* = \int_{V_F} \rho_F e^* dV_F = \int_{V_o} \rho_o e^* dV_o \qquad (2.51)$$

and e^* is the specific energy of the body. Substitution of (2.29), (2.32), and (2.51) into (2.50) yields

$$\int_{V_o} \rho_o \dot{e}^* dV_o = \int_{S_o} (T_i + T_i^o)\dot{u}_i dS_o + \int_{V_o} \rho_o (f_i + f_i^o)\dot{u}_i dV_o \qquad (2.52)$$

The equilibrium and boundary conditions for the initial state together with (2.38c) may be used to reduce the right hand side of (2.52) to

$$\int_{V_o} \rho_o \dot{e}^* dV_o = \int_{V_o} (\rho_o f_i \dot{u}_i + \sigma_{ij}^o \dot{u}_{i,j}) dV_o - \int_{S_o} s_j n_j^o dS_o \qquad (2.53)$$

where

$$s_j = -D^o_{ijk\ell} u_{k,\ell} \dot{u}_i \qquad (2.54)$$

The quantity s_j is called the energy flux vector since it is seen to represent the energy flow per unit area, per unit time out of the surface S_o. The application of Gauss' Theorem to the surface integral in (2.53) then yields the energy continuity equation,

$$\rho_o \dot{e}^* = \rho_o f_i \dot{u}_i + t^o_{ij} \dot{e}_{ij} - s_{j,j} \qquad (2.55)$$

Stated in words, the local rate of increase of energy is equal to the rate at which the body force and initial stress do work minus the local rate of efflux of energy.

3. FORCED MOTION OF AN INITIALLY STRESSED SOLID

One of the central problems arising in engineering applications of the theory developed in Section 2 is the forced, incremental motion of a bounded, initially stressed solid with time dependent, incremental forcing functions and boundary conditions. We shall resolve this problem in the manner of reference [27] using classical mathematical techniques.

With reference to the development in Section 2, a well posed forced motion problem can be stated as follows.

Equation of incremental motion:

$$\sigma_{ij,j} + \rho_o f_i = \rho_o \ddot{u}_i \text{ in } V_o \text{ for all } t > 0 \qquad (3.1a)$$

where

$$\sigma_{ij} = \tau_{ij} + t^o_{ij} u_{k,k} - t^o_{i\ell} u_{j,\ell} = D^o_{ijk\ell} u_{k,\ell} \qquad (3.1b)$$

and

$$D^o_{ijk\ell} = D^o_{k\ell ij} \qquad (3.1c)$$

Boundary conditions:

$$u_i = F_i(x,t) \text{ on } S_{o1} \tag{3.1d}$$

$$T_i = \sigma_{ij} n_j^o = G_i(x,t) \text{ on } S_{o2} \tag{3.1e}$$

where $S_o = S_{o1} + S_{o2}$ is the total surface of the initially stressed solid in the initial configuration.

Initial conditions:

$$u_i(x,o) = u_i^{(o)}(x), \quad \dot{u}_i(x,o) = \dot{u}_i^{(o)}(x) \tag{3.1f}$$

throughout V_o at $t=0$.

It can be shown that the solution of the problem as characterized by equations (3.1) is unique if (2.44) is satisfied (see [3], p. 256).

To obtain a solution of (3.1), we proceed in a manner similar to [27], i.e., we assume a solution of (3.1) in the form

$$u_i(x,t) = u_i^{(s)}(x,t) + \sum_{m=1}^{\infty} U_i^{(m)}(x) \cdot q_m(t) \tag{3.2}$$

The "quasi-static" part $u_i^{(s)}(x,t)$ of the solution satisfies equations (3.1a) through (3.1e) with inertia terms in (3.1a deleted, i.e.,

$$\sigma_{ij,j}^{(s)} + \rho_o f_i = 0 \text{ in } V_o \text{ for all } t>0 \tag{3.3a}$$

$$\sigma_{ij}^{(s)} = \tau_{ij}^{(s)} + \tau_{ij}^{o} u_{k,k}^{(s)} - \tau_{i\ell}^{o} u_{j,\ell}^{(s)} = D_{ijk\ell}^{o} u_{k,\ell}^{(s)} \tag{3.3b}$$

$$u_i^{(s)} = F_i(x,t) \text{ on } S_{o1} \tag{3.3c}$$

$$T_i^{(s)} = \sigma_{ij}^{(s)} n_j^o = G_i(x,t) \text{ on } S_o \tag{3.3d}$$

The eigenfunctions $U_i^{(m)}(x)$ of the associated homogeneous problem are characterized by the equations

$$\sigma_{ij,j}^{(m)} + \rho_o \omega_m^2 U_i^{(m)} = 0 \text{ in } V_o \tag{5.4a}$$

$$\sigma_{ij}^{(m)} = D_{ijk\ell}^o U_{k,\ell}^{(m)} \tag{5.4b}$$

$$U_i^{(m)} = 0 \text{ on } S_u \tag{5.4c}$$

$$F_i^{(m)} = \sigma_{ij}^{(m)} n_j^o = 0 \text{ on } S_o. \tag{5.4d}$$

In the following we shall assume that the criterion for infinitesimal stability (2.11) is satisfied for all $t \geq 0$. In this case (see Section 2), the eigenvalues ω_m^2 are real, positive numbers. If, in addition, we impose the strict inequality $H > 0$ (super-stability, [5]), we can conclude that no eigenvalue can vanish. Thus we shall rule out the trivial solution $U_i^{(m)} = 0$ and the rigid body displacement field $\vec{U}^{(m)} = \vec{a} + \vec{b} \times \vec{R}_o$, where $\vec{a}$ and $\vec{b}$ are constant vectors and $\vec{R}_o$ is the position vector in V_o. To each eigenfunction $U_i^{(m)}$ there corresponds an eigenvalue ω_m^2, where ω_m is a natural frequency of vibration. We shall assume that the characterization of the eigenvalue problem (5.4) results in a denumerable set of solutions $\{U_i^{(m)}, \omega_m^2;\ m=1,2,3,\ldots\}$. If the rigid body motion is not an admissible eigenfunction, the case of a vanishing eigenvalue does not arise, and the eigenvalues can be ordered as follows: $0 < \omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \leq \ldots$ If there are no degeneracies (i.e., if the eignevalues are distinct), the eigenfunctions can be shown to be orthogonal. If degeneracies occur, i.e., if two or more different eigenfunctions correspond to the same eigenvalue, each set of degenerate eigenfuctions can be orthogonalized by the Gram-Schmidt process. In either case we have (see (2.15))

$$\int_{V_o} \rho_o U_i^{(m)} U_i^{(n)} dV_o = \delta_{mn} \tag{5.5}$$

where δ_{mn} is the Kronecker delta symbol. Equation (3.5) also implies that the eigenfunctions $U_i^{(m)}$ have been suitably normalized.

In view of (3.1d), (3.1e), (3.3c), (3.3d), (3.4c), and (3.4d) the assumed solution (3.2) satisfies the *non-homogeneous* boundary conditions (3.1d) and (3.1e). To determine the scalar functions $q_m(t)$ we now substitute (3.2) into (3.1a) and (3.1b). If, in addition, we apply (3.3a), (3.3b), (3.4a), and (3.4b), we obtain

$$\sum_{m=1}^{\infty} \rho_o U_i^{(m)} (\ddot{q}_m + \omega_m^2 q_m) = - \rho_o \ddot{u}_i^{(s)} \tag{3.6a}$$

Similarly, if we set t=0 in (3.2) and substitute the resulting equation into (3.1f) we readily obtain

$$\sum_{m=1}^{\infty} U_i^{(m)}(x)\cdot q_m(o) = u_i^{(o)}(x) - u_i^{(s)}(x,o) \tag{3.6b}$$

$$\sum_{m=1}^{\infty} U_i^{(m)}(x)\cdot \dot{q}_m(o) = \dot{u}_i^{(o)}(x) - \dot{u}_i^{(s)}(x,o) \tag{3.6c}$$

Now we multiply (3.6a), (3.6b), and (3.6c) by $U_i^{(n)}$, $\rho_o U_i^{(n)}$, and $\rho_o U_i^{(n)}$, respectively, and integrate the resulting equations over the volume V_o. Upon application of (3.5) we obtain

$$\ddot{q}_m + \omega_m^2 q_m = \ddot{Q}_m(t) \text{ for } t>0 \tag{3.7a}$$

$$q_m(o) - Q_m(o) = \int_{V_o} \rho_o U_i^{(m)}(x)\cdot u_i^{(o)}(x) dV_o \tag{3.7b}$$

$$\dot{q}_m(o) - \dot{Q}_m(o) = \int_{V_o} \rho_o U_i^{(m)}(x)\cdot \dot{u}_i^{(o)}(x) dV_o \tag{3.7c}$$

where

$$Q_m(t) = - \int_{V_o} \rho_o u_i^{(s)} U_i^{(m)} dV_o \tag{3.8}$$

A more convenient form of $Q_m(t)$ can be obtained with the aid of (3.4a) and (3.4b) as follows:

$$\omega_m^2 Q_m(t) = -\int_{V_o} \rho_o \omega_m^2 U_i^{(m)} u_i^{(s)} dV_o = \int_{V_o} \sigma_{ij,j}^{(m)} u_i^{(s)} dV_o$$

$$= \int_{S_{o1}} T_i^{(m)} F_i dS_o - \int_{V_o} \sigma_{ij}^{(m)} u_{i,j}^{(s)} dV_o$$

But $\sigma_{ij}^{(m)} u_{i,j}^{(s)} = \sigma_{ij}^{(s)} U_{i,j}^{(m)}$ because of (3.1c), and therefore

$$\int_{V_o} \sigma_{ij}^{(m)} u_{i,j}^{(s)} dV_o = \int_{V_o} \sigma_{ij}^{(s)} U_{i,j}^{(m)} dV_o = \int_{V_o} (\sigma_{ij}^{(s)} U_i^{(m)})_{,j} dV_o - \int_{V_o} U_i^{(m)} \sigma_{ij,j}^{(s)} dV_o$$

$$= \int_{S_{o2}} T_i^{(s)} U_i^{(m)} dS_o - \int_{V_o} U_i^{(m)} (D^o_{ijkl} u^{(s)}_{k,l})_{,j} dV_o$$

$$= \int_{S_{o2}} G_i U_i^{(m)} dS_o + \int_{V_o} \rho_o f_i U_i^{(m)} dV_o$$

where we have used the integral theorem of Green/Gauss/Ostrogradskii and (3.3d), (3.3b), and (3.3a). Consequently we can write

$$\omega_m^2 Q_m(t) = -\int_{V_o} \rho_o f_i U_i^{(m)} dV_o + \int_{S_{o1}} T_i^{(m)} F_i dS_o - \int_{S_{o2}} U_i^{(m)} G_i dS_o \qquad (3.9)$$

The solution of (3.7a) is

$$q_m(t) = [q_m(o) - Q_m(o)]\cos \omega_m t + \frac{1}{\omega_m} [\dot{q}_m(o) - \dot{Q}_m(o)]\sin \omega_m t$$

$$+ Q_m(t) - \omega_m \int_o^t Q_m(\xi) \sin \omega_m (t-\xi) d\xi \qquad (3.10)$$

Thus the formal solution of the non-homogeneous problem posed by (3.1) is given by (3.2), where $u_i^{(s)}$ satisfies (3.3), $\{U_i^{(m)}, \omega_m;$ $m=1,2,3,\ldots\}$ is the complete solution set of (3.4) which also satisfies (3.5), and the scalar function $q_{ij}(t)$ is given by (3.1) in conjunction with (3.9), (3.7b) and (3.7c).

4. THE SOLID UNDER INITIAL HYDROSTATIC PRESSURE

To illustrate the application of the general theory to a specific example we shall examine the case of a homogeneous, isotropic body subject to an initial hydrostatic pressure. This problem has received considerable attention because of its applications to solid state physics and to geophysics. Some of the relevant references are [5], [8-11]. In this case,

$$\sigma^o_{ij} = -p_o\delta_{ij} \tag{4.1a}$$

$$x^o_i = (1-2\epsilon_o)^{1/2} a_i \tag{4.1b}$$

$$J_o = \det\left(\frac{\partial x^o_i}{\partial a_j}\right) = (1-2\epsilon_o)^{3/2} \tag{4.1c}$$

$$E^o_{ij} = -\epsilon_o\delta_{ij} \tag{4.1d}$$

$$\rho_a = \rho_o J_o = \rho_o(1-2\epsilon_o)^{3/2} \tag{4.1e}$$

$$\sigma^o_{ij} = J_o^{-1}\frac{\partial x^o_i}{\partial a_k}\frac{\partial x^o_j}{\partial a_l} t^o_{kl} = (1-2\epsilon_o)^{-1/2} t^o_{ij} \tag{4.1f}$$

In view of (4.1a) and (2.19), the incremental stress-strain law becomes

$$\tau_{ij} = S^o_{ijkl} e_{kl} \tag{4.2a}$$

where

$$S^o_{ijkl} = C^o_{ijkl} = B^o_{ijkl} + p_o\delta_{ij}\delta_{kl} - p_o(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.2b}$$

and

$$B^o_{ijk} = 0 \qquad (4.2)$$

Furthermore, (4.1b), (4.1c), and (2.11d) yield the result,

$$B'_{ijkl} = (1-2\eta_o)^{-3/2} A^o_{ijkl} \qquad (4.3)$$

where

$$A^o_{ijkl} = \left(\frac{\partial^2 W'}{\partial \eta_{ij}\,\partial \eta_{kl}}\right)_o$$

At this point we could continue at a fairly general level by assuming that the strain energy density is only a function of the three principal invariants of the strain, η_{ij}, without actually specifying the precise form of W'. However, in order to more clearly illustrate the theory we will specify a functional form for W'. The choice of this form is motivated by the following question. Assuming that the initial deformation is relatively small ($\eta_o \ll 1$), what is the form of the most general linear correction term which must be applied to the linear elasticities and wave speeds measured from the reference state in order to obtain the corresponding quantities measured in the initial state? We shall presently show that it suffices to retain terms up to and including the third order in a Taylor series expansion of W' about the natural state to answer this question. Thus, we assume that

$$W' = \frac{1}{2} A^a_{ijkl}\eta_{ij}\eta_{kl} + \frac{1}{6} A^a_{ijklmn}\eta_{ij}\eta_{kl}\eta_{mn} \qquad (4.4)$$

For this function,

$$T^o_{ij} = A^a_{ijkl}\eta^o_{kl} + \frac{1}{2} A^a_{ijklmn}\eta^o_{kl}\eta^o_{mn} \qquad (4.5)$$

and

$$A^o_{ijkl} = A^a_{ijkl} + A^a_{ijklmn}\eta^o_{mn} \qquad (4.6)$$

where the superscript a refers to the natural state. Equation (4.6) verifies our assertion that only terms up to the third order (or second order elasticities) are needed to obtain the linear correction term for $A^a_{ijk\ell}$. Substituting (4.1d) into (4.5) and (4.6) yields the result

$$T^o_{ij} = -\epsilon_o A^a_{ijkk} + \frac{1}{2}\epsilon_o^2 A^a_{ijkkmm} \tag{4.7a}$$

$$A^o_{ijk\ell} = A^a_{ijk\ell} - \epsilon_o A^a_{ijk\ell mm} \tag{4.7b}$$

For a homogeneous, isotropic medium,

$$A^a_{ijk\ell} = \lambda_a \delta_{ij}\delta_{k\ell} + \mu_a(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{4.8a}$$

$$\begin{aligned} A^a_{ijk\ell mn} = {} & \nu^a_1 \delta_{ij}\delta_{k\ell}\delta_{mn} + \nu^a_2 [\delta_{ij}(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m}) \\ & + \delta_{k\ell}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{mn}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})] \\ & + \nu^a_3 [\delta_{ik}(\delta_{jm}\delta_{\ell n} + \delta_{jn}\delta_{\ell m}) + \delta_{j\ell}(\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}) \\ & + \delta_{i\ell}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) + \delta_{jk}(\delta_{im}\delta_{\ell n} + \delta_{in}\delta_{\ell m})] \end{aligned} \tag{4.8b}$$

where (λ_a, μ_a) and $(\nu^a_1, \nu^a_2, \nu^a_3)$ are the Lamé parameters of first and second order elasticity *measured from the natural state*. Here we are adopting the notation of Toupin and Bernstein [5]. Substitution of (4.1a) and (4.7a) into (4.1f) yields the relation

$$p_o \delta_{ij} = (1-2\epsilon_o)^{-1/2} [\epsilon_o A^a_{ijkk} - \frac{1}{2}\epsilon_o^2 A^a_{ijkkmm}]$$

and since we are seeking a linear correction, we may drop the ϵ_o^2 term to obtain

$$p_o \delta_{ij} = \epsilon_o A^a_{ijkk} = (3\lambda_a + 2\mu_a)\epsilon_o \delta_{ij}$$

or

$$\epsilon_o = \frac{p_o}{3\lambda_a+2\mu_a} = \frac{p_o}{3K_a} \tag{4.9}$$

where $K_a = \lambda_a + \frac{2}{3}\mu_a$ is the bulk modulus measured in the natural state. Further substitution of (4.8b) into (4.7b) yields

$$A^o_{ijkl} = [\lambda_a - \epsilon_o(3\nu_1^a + 4\nu_2^a)]\,\delta_{ij}\delta_{kl} + [\mu_a - \epsilon_o(3\nu_2^a + 4\nu_3^a)]\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.10}$$

and further substitution into (4.3), using (4.9) yields, after linearizing with respect to ϵ_o,

$$B^o_{ijkl} = [\lambda_a - \frac{p_o}{3K_a}(\lambda_a+3\nu_1^a+4\nu_2^a)]\,\delta_{ij}\delta_{kl} + [\mu_a - \frac{p_o}{3K_a}(\mu_a+3\nu_2^a+4\nu_3^a)]\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{4.11}$$

Note that the linear correction term has now been put in terms of the applied pressure p_o. The substitution of (4.11) into (4.2) now yields the incremental stress strain law,

$$t_{ij} = \lambda_o e\delta_{ij} + 2\mu_o e_{ij} \tag{4.12}$$

where,

$$\lambda_o = \lambda_a + \frac{p_o}{3K_a}(2\lambda_a+2\mu_a-3\nu_1^a-4\nu_2^a)$$

$$\mu_o = \mu_a - \frac{p_o}{3K_a}(3\lambda_a+3\mu_a+3\nu_2^a+4\nu_3^a) \tag{4.13}$$

The quantities λ_o and μ_o are usually referred to as the apparent elasticities of the pre-stressed body. Similarly, we may define an apparent bulk modulus from the relation

$$t_{kk} = 3K_o e_{kk} = (3\lambda_o+2\mu_o)e_{kk}$$

thus,

$$3K_o = 3\lambda_o + 2\mu_o$$
$$= 3K_a\left[1 - \frac{p_o}{3K_a}\left(\frac{9\nu_1^a + 18\nu_2^a + 8\nu_3^a}{3K_a}\right)\right] \qquad (4.14)$$

Similarly, apparent values for Young's modulus and Poisson's ratio are obtained from

$$E_o = \frac{3\mu_o K_o}{\lambda_o + \mu_o} \quad \text{and} \quad \nu_o = \frac{E_o \lambda_o}{2\mu_o 3K_o}$$

which, in view of (4.13) and (4.14) yields,

$$E_o = E_a\left[1 - \frac{p_o}{\mu_a}\left(1 + \frac{\mu_a + 3\nu_2^a + 4\nu_3^a}{3K_a}\right)\right] \qquad (4.15a)$$

$$\nu_o = \nu_a\left[1 + \frac{p_o}{\lambda_a}(1-\alpha)\right] \qquad (4.15b)$$

where

$$\alpha = \frac{\mu_a(3\nu_1^a + 4\nu_2^a) - \lambda_a(3\nu_2^a + 4\nu_3^a)}{(\lambda_a + \mu_a)(3\lambda_a + 2\mu_a)} \qquad (4.15c)$$

In view of (2.25) and (4.13),

$$D^o_{ijkl} = (\lambda_o - p_o)\delta_{ij}\delta_{kl} + \mu_o\delta_{ik}\delta_{jl} + (\mu_o + p_o)\delta_{il}\delta_{jk} \qquad (4.16)$$

Substitution of (4.16) into (2.38) then yields the equation for the incremental motion,

$$(\lambda_o + \mu_o)u_{j,ji} + \mu_o u_{i,jj} + \rho_o f_i = \rho_o \ddot{u}_i \qquad (4.17a)$$

and the associated natural boundary conditions

$$(\lambda_o - p_o)u_{k,k}n_i^o + \mu_o u_{i,j}n_j^o + (\mu_o + p_o)u_{j,i}n_j^o = T_i \qquad (4.17b)$$

From (4.17a) one can readily show that, for an unbounded medium free of body forces, two solutions of the form

[illegible]

$$u_i = \mathrm{Re}\, U_i e^{ik(x_j n_j^o - V^o t)} \tag{4.18}$$

exist corresponding to the two wave speeds

$$\rho_o (V_L^o)^2 = \lambda_o + 2\mu_o$$
$$\rho_o (V_T^o)^2 = \mu_o \tag{4.19}$$

or, in view of (4.15) and (4.16),

$$(V_L^o)^2 = \frac{\lambda_a + 2\mu_a}{\rho_a}\left[1 - \frac{p_o}{3K_a}\left(\frac{7\lambda_a + 10\mu_a + 3\nu_1^a + 10\nu_2^a + 8\nu_3^a}{\lambda_a + 2\mu_a}\right)\right]$$
$$(V_T^o)^2 = \frac{\mu_a}{\rho_a}\left[1 - \frac{p_o}{3K_a}\left(\frac{3\lambda_a + 6\mu_a + 3\nu_2^a + 4\nu_3^a}{\mu_a}\right)\right] \tag{4.20}$$

This same result is also obtained by Toupin and Bernstein [3], as well as by Thurston and Brugger [9]. In an earlier paper, Birch [15], derived a restricted form of (4.20) based on the assumption that W* is a quadratic function of the Eulerian strains, rather than the Lagrangian strains. In terms of strain invariants, Birch assumed that

$$W^* = \left(\frac{\lambda + 2\mu}{2}\right) I_e^2 - 2\mu II_e \tag{4.21}$$

These are related to the principal invariants of the Lagrangian strain tensor as follows:

$$I_e = \frac{I_E + 4II_E + 12III_E}{1 + 2I_E + 4II_E + 8III_E}$$
$$II_e = \frac{II_E + 6III_E}{1 + 2I_E + 4II_E + 8III_E} \tag{4.22}$$

where

$$I_E = E_{ii}$$

$$II_E = \frac{1}{2}(I_E^2 - E_{ij}E_{ji})$$

$$III_E = \det(E_{ij}) = \frac{1}{6}(I_E^3 - 3I_E E_{ij}E_{ji} + 2E_{ij}E_{jk}E_{ki})$$

The substitution of (4.22) into (4.21) yields, after neglecting terms higher than the third order,

$$W^* = (\frac{\lambda+2\mu}{2})I_E^2 - 2\mu II_E + 4(\lambda+3\mu)I_E II_E - 2(\lambda+2\mu)I_E^3 - 12\mu III_E \quad (4.23a)$$

On the other hand, substitution of (4.8) into (4.4) yields the result

$$W^* = (\frac{\lambda+2\mu}{2})I_E^2 - 2\mu II_E - 2(\nu_2+2\nu_3)I_E II_E$$
$$+ \frac{1}{6}(\nu_1+6\nu_2+8\nu_3)I_E^3 + 4\nu_3 III_E \quad (4.23b)$$

Birch, has therefore assumed, instead of three independent third order constants, the values

$$\nu_1 = 0, \qquad \nu_2 = -2\lambda, \qquad \nu_3 = -3\mu \quad (4.24)$$

These values, when substituted into (4.20) yield Brich's results, e.g.,

$$(V_L^o)^2 = \left(\frac{\lambda^a+2\mu^a}{\rho_a}\right)\left[1 + \frac{P_o}{3K_a}\left(\frac{13\lambda^a+14\mu^a}{\lambda^a+2\mu^a}\right)\right]$$
$$(V_T^o)^2 = \left(\frac{\mu_a}{\rho_a}\right)\left[1 + \frac{P_o}{3K_a}\left(\frac{3\lambda^a+6\mu^a}{\mu^a}\right)\right] \quad (4.25)$$

These formulae are very appealing since they contain only the first order elasticities λ^a and μ^a and they also predict that the wave speeds increase with pressure as one might expect. However, recently Soga and Anderson [15], have shown experimentally that for isotropic, polycrystalline ZnO specimens, the shear wave speed decreases with increasing pressure. This result can not be predicted using Birch's formulae (4.25). However, the general result (4.20) can predict either an increase or a decrease in wave speed depending on the values of the second-order elasticities (third-order elastic constants) ν_1^a, ν_2^a, ν_3^a. In a later paper [14], Birch does adopt the Lagrangian viewpoint and include the second-order elasticities in his study of cubic crystals. Unfortunately, many authors still use the earlier results.

A great deal of literature exists on the determination of the second-order elasticities for crystals. We wish to point out again that these are also commonly called the third-order elastic constants. References [16-20] contain summaries of the available data and give extensive bibliographies to the existing literature. The isotropic elastic constants of polycrystalline aggregates can be computed from single-crystal data using averaging techniques suggested by Voigt, Reuss, and Hill. Anderson [21] has applied these techniques to compute the first-order elasticities of several substances. Hamilton and Parrott [22], Cousins [23], and Barsch [24], have further applied these methods to determine the second-order elasticities of quasi-isotropic materials. However, the values computed by these averaging techniques are approximate in that they ignore the complexities introduced by voids, inclusions, and grain boundaries which occur in real materials. Ledbetter and Naimon [25] have recently suggested a new averaging technique based on the equivalence of the Debye temperature for single crystals and polycrystals of the same material. Although Toupin and Bernstein [5] have outlined a series of five independent experiments which can be used to determine the isotropic elasticities, λ^a, μ^a, ν_1^a, ν_2^a, ν_3^a, very little experimental data is available on the values of the second order elasticities for isotropic materials. Smith, Stern and Stephens [26] reported values of the first and second order elasticities obtained by the ultrasonic pulse-echo method for five steel alloys, five aluminum alloys, magnesium tooling plate, molybdenum, and tungsten. The values given in Table 1.1 are taken from that reference. Using the data in Table 1.1, we can

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

compute the pressure derivatives of the wave speeds and the various elastic constants. These are presented in Table 1.2. To simplify the computation we introduce the dimensionless quantities:

$$\frac{[illegible]_o}{[illegible]_a}, \quad \frac{\mu_o}{\mu_a}, \quad K = \frac{K_o}{K_a}, \quad E = \frac{E_o}{E_a}, \quad \frac{[illegible]_o}{[illegible]_a},$$

$$V_L = \frac{V_L^o}{V_L^a}, \quad V_T = \frac{V_T^o}{V_T^a}, \quad \text{and} \quad p = \frac{p_o}{[illegible]K_a}$$

Table 4.[illegible]

	$\frac{d\lambda}{dp}$	$\frac{d\mu}{dp}$	$\frac{dK}{dp}$	$\frac{dE}{dp}$	$\frac{d\nu}{dp}$	$\frac{dV_L}{dp}$	$\frac{dV_T}{dp}$	[illegible] V_L	[illegible] V_T
Carbon Steel	[illegible]3.3	12.1	1[illegible]	1[illegible]	4.9	6.7	4.4	4.[illegible]	[illegible]
Nickel Steel	1[illegible].1	1[illegible].1	11.8	1[illegible]	[illegible]	4.[illegible]	[illegible]	4.[illegible]	[illegible]
Aluminum	[illegible]8.8	[illegible]	26.4	2[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Magnesium	1[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4.[illegible]	4.[illegible]	[illegible]
Molybdenum	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4.[illegible]	[illegible]
Tungsten	1[illegible]	[illegible]	1[illegible]	[illegible]	[illegible]	4.[illegible]	[illegible]	4.[illegible]	4.[illegible]

Then, (4.13), (4.14), (4.15), and (4.20) yield the results

$$\frac{d\lambda}{dp} = (2\lambda_a+2\mu_a-3\nu_1^a-4\nu_2^a)/\lambda_a$$

$$\frac{d\mu}{dp} = -(3\lambda_a+3\mu_a+3\nu_2^a+4\nu_3^a)/\mu_a$$

$$\frac{dK}{dp} = -(9\nu_1^a+18\nu_2^a+8\nu_3^a)/(3K_a)$$

$$\frac{dE}{dp} = -(3K_a+6\mu_a+3\nu_2^a+4\nu_3^a)/\mu_a$$

$$\frac{d\nu}{dp} = 3K_a(1-[illegible])/[illegible]_a$$

$$\frac{dV_L}{dp} = -(7\lambda_a+10\mu_a+3\nu_1^a+10\nu_2^a+8\nu_3^a)/(2\lambda_a+4\mu_a)$$

$$\frac{dV_T}{dp} = -(3\lambda_a+6\mu_a+3\nu_2^a+4\nu_3^a)/(12\mu_a)$$

With the exception of Poisson's ratio for molybdenum, all the pressure derivatives in Table 4.2 are positive, and indeed, the one exception is so small that it could well fall within the limits of the experimental error of the original data. Furthermore, it is noted that all the dimensionless pressure derivatives of Young's modulus are of the order of magnitude of 10, i.e., $\frac{dE}{dp} \approx 10$. Therefore, with reference to (4.15a),

$$\frac{E_o}{E_a} = 1 + \left(\frac{dE}{dp}\right)p \approx 1 + 10p = 1 + \frac{10p_o}{3K_a}$$

Since $3K_a \approx 50\times10^{10}$ Pa (see Table 4.1), we conclude that $\frac{E_o}{E_a} \approx 1 + (.2\times10^{-10})p_o$ if p_o is measured in Pa, or, $\frac{E_o}{E_a} \approx 1 + (1.4\times10^{-7})p_o$ if p_o is measured in lb./in.^{2}. Thus, a pressure of 100,000 lb/in^2 is required to cause a 1.4% increase in the Young's modulus of any of the metals listed in Table 4.1. A similar result is seen to apply to any of the other properties listed in this table. Thus, in most engineering applications it should be possible to ignore the effects on the elastic constants of an initial hydrostatic pressure less than 10^4 lb/in^2. In fact, since any initial hydrostatic pressure causes a stable equilibrium configuration, the effects of the pressure may be completely ignored in such cases. The stability of the initial configuration follows from (2.45), (2.46) and (4.16). For example, substitution of (4.16) into (2.45) yields:

$$p^o_{ijkl}\xi_i\eta_k\xi_j\eta_l = \mu_o\xi^2\eta^2 + (\lambda_o+\mu_o)(\xi_i\eta_i)^2$$

$$\geq \mu_o\xi^2\eta^2 \text{ if } \lambda_o+\mu_o \geq 0$$

Thus, the initial configuration satisfies the Hadamard infinitesimal stability criterion. Similarly, substitution of (4.16) into (2.46) yields:

$$D^{o}_{ijk\ell}\epsilon_{ij}\epsilon_{k\ell} = (2\mu_o+p_o)\epsilon_{ij}\epsilon_{ij} + (\lambda_o-p_o)\epsilon_{ii}\epsilon_{jj}$$

$$\geq (2\mu_o+p_o)\epsilon_{ij}\epsilon_{ij} \quad \text{if } \lambda_o \geq p_o$$

Thus, the initial configuration also satisfies the Liapounov stability criterion.

5. THE INITIALLY STRESSED BEAM

5.1 Basic Equations

In this section we consider the incremental motion of an initially stressed beam. We shall assume that the cross-section of the beam has at least one plane of symmetry, which we shall take as the x-z plane of our Cartesian axis system, where x is the longitudinal axis of the beam. It is assumed that the beam is subjected to an initial axial stress $\sigma=\sigma(x)$, with all other pre-stress tensor components vanishing, i.e.,

$$\begin{bmatrix} \sigma^{o}_{11} & \sigma^{o}_{12} & \sigma^{o}_{13} \\ \sigma^{o}_{21} & \sigma^{o}_{22} & \sigma^{o}_{23} \\ \sigma^{o}_{31} & \sigma^{o}_{32} & \sigma^{o}_{33} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.1}$$

The incremental displacements are taken to be

$$u_1 = z\psi(x,t), \quad u_2 = 0, \quad u_3 = w(x,t) \tag{5.2}$$

and in view of (5.1), (5.2), and (3.1b), we obtain

$$e_{11} = \epsilon_{11}, \quad e_{13} = \epsilon_{13} - \sigma\frac{\partial w}{\partial x}, \quad e_{31} = \epsilon_{31} = \epsilon_{13} \tag{5.3}$$

Upon substitution of (5.2) into (2.22b), using (3.1b) and (5.3), we readily obtain the first variation of the strain energy in the beam in the form

$$\delta U = \int_{V_o} (z\sigma_{11} \frac{\partial}{\partial x} \delta\varphi + \sigma_{13}\delta\varphi + \tau_{13} \frac{\partial}{\partial x} \delta w) dV_o \qquad (5.4)$$

We define the stress resultants as

$$M = \int_{A_o} z\,\sigma_{11}\, dA_o \;,\; V = \int_{A_o} \tau_{13}\, dA_o = Q + A_o\,\sigma \frac{\partial w}{\partial x} \qquad (5.5)$$

where $Q = \int_{A_o} \sigma_{13}\, dA_o$, and observe that $\int_{V_o} \ldots\ldots dV_o = \int_{L_1}^{L_2} \int_{A_o} \ldots\ldots dA_o dx$. With these definitions, equation (5.4) assumes the form

$$\delta U = \int_{L_1}^{L_2} (M \frac{\partial}{\partial x} \delta\varphi + Q\delta\varphi + V \frac{\partial}{\partial x} \delta w)\, dx$$

Upon integration by parts, we obtain

$$\delta U = \int_{L_1}^{L_2} [(- \frac{\partial M}{\partial x} + Q)\,\delta\varphi - \frac{\partial V}{\partial x} \delta w]\, dx + (M\,\delta\varphi + V\,\delta w)\Big|_{L_1}^{L_2} \qquad (5.6)$$

The time integral of the first variation of the (incremental) kinetic energy is (see 2.27).

$$\int_{t_1}^{t_2} \delta T\, dt = - \int_{t_1}^{t_2} \int_{V_o} \ddot{u}_i\, \delta u_i\, \rho_o dV_o dt =$$

$$= - \int_{t_1}^{t_2} \int_{V_o} (z^2 \ddot{\varphi}\delta\varphi + \ddot{w}\delta w)\rho_o dV_o dt$$

$$= - \int_{t_1}^{t_2} \int_{L_1}^{L_2} (\rho_o I_o \ddot{\varphi}\delta\varphi + \rho_o A_o \ddot{w}\delta w) dx dt \qquad (5.7)$$

where $I_o = \int_{A_o} z^2 dA_o$, $A_o = \int_{A_o} dA_o$

To find the virtual work of external, incremental forces we use (2.55):

$$\delta W = \int_{S_o} T_i \delta u_i \, dS_o = \int_{S_o} \vec{T} \cdot \delta \vec{u} \, dS_o$$

$$= \int_{\Sigma_o} \vec{T} \cdot \delta \vec{u} \, d\Sigma_o + \int_{A_{o1}} \vec{T} \cdot \delta \vec{u} \, dA_o + \int_{A_{o2}} \vec{T} \cdot \delta \vec{u} \, dA_o$$

where $S_o = \Sigma_o + A_{o1} + A_{o2}$. In this expression A_{o1} and A_{o2} denote the magnitude of the areas of the plane cross section A_o of the initially stressed beam at $x=L_1$ and $x=L_2$, respectively, while Σ_o is the surface area of the beam for $L_1 < x < L_2$. Observing the usual sign conventions we have

$$\text{on } \Sigma_o : \quad \vec{T} = \vec{i} T_1 + \vec{k} T_3$$

$$\text{on } A_{o1} : \quad \vec{T} = -\vec{i} F_1 - \vec{k} F_3$$

$$\text{on } A_{o2} : \quad \vec{T} = \vec{i} F_1 + \vec{k} F_3$$

and on Σ_o, A_{o1} and A_{o2} we have $\delta \vec{u} = \vec{i} \delta u_1 + \vec{k} \delta u_3$. Consequently, using (5.2), we obtain

$$\delta W = \int_{L_1}^{L_2} (p \delta w + m \delta \psi) \, dx + (M \delta \psi + V \delta w) \Big|_{L_1}^{L_2} \qquad (5.8)$$

where

$$\int_{\Sigma_o} T_1 z \, dS_o = \int_{L_1}^{L_2} \oint_{C_o} T_1 z \, dC_o \, dx = \int_{L_1}^{L_2} m \, dx$$

$$\int_{\Sigma_o} T_3 \, dS_o = \int_{L_1}^{L_2} \oint_{C_o} T_3 \, dC_o \, dx = \int_{L_1}^{L_2} p \, dx$$

and where we define

$$\int_{C_o} T_{12} z \, dC_o = m, \qquad \int_{C_o} T_{12} dC_o = p$$

$$\left.\begin{aligned} M^*(L_i) &= \int_{A_{oi}} T_{11} z \, dA_o \\ V^*(L_i) &= \int_{A_{oi}} T_{12} dA_o \end{aligned}\right\} \quad i = 1,2.$$

If we now invoke Hamilton's principle for incremental motions (2.57), we readily obtain the stress equations of incremental motion and the associated, admissible boundary conditions

$$-\rho_o I_o \ddot{\psi} + \frac{\partial M}{\partial x} - Q + m = 0 \qquad (5.9a)$$

$$-\rho_o A_o \ddot{w} + \frac{\partial V}{\partial x} + p = 0 \qquad \text{for } L_1 < x < L_2 \qquad (5.9b)$$

$$\left[(M^*-M)\delta\psi + (V^*-V)\delta w\right]_{L_1}^{L_2} = 0 \qquad (5.9c)$$

Equations (5.9) characterize the initially stressed Timoshenko beam model [7]. To obtain the initially stressed Euler-Bernoulli beam model, we set $\rho_o I_o \ddot{\psi} = m = 0$, and let $\psi = -\frac{\partial w}{\partial x}$. In this case equations (5.9) reduce to

$$Q = \frac{\partial M}{\partial x} \qquad (5.10a)$$

$$-\rho_o A_o \ddot{w} + \frac{\partial Q}{\partial x} + A_o \sigma \frac{\partial^2 w}{\partial x^2} + p = 0 \qquad \text{for } L_1 < x < L_2 \qquad (5.10b)$$

$$\left[-(M^*-M)\delta\left(\frac{\partial w}{\partial x}\right) + (V^*-V)\delta w\right]_{L_1}^{L_2} = 0 \qquad (5.10c)$$

If the beam is unable to sustain a bending moment we have $M = 0$, $M^* = 0$, and equations (5.10) reduce to

$$-\rho_o A_o \ddot{w} + A_o \sigma \frac{\partial^2 w}{\partial x^2} + p = 0 \qquad \text{for } L_1 < x < L_2 \qquad (5.11a)$$

[illegible] (5.11a)

$$[(V^*-V)\delta w]_{L_2}^{L_1} = 0 \qquad (5.11b)$$

where $V = A_o \sigma \frac{\partial w}{\partial x}$. Equations (5.11) characterize the well known, classical model of the vibrating string.

We now derive the stress-displacement relations corresponding to the beam model characterized by (5.9). As a point of departure, we postulate a hyperelastic medium with a constant, uniaxial, initial stress field σ in the x_1 direction. (See (5.1)). In this case we have $x_1^o = (1+\epsilon_o)a_1$, $x_2^o = (1-\nu_a\epsilon_o)a_2$, $x_3^o = (1-\nu_a\epsilon_o)a_3$, and according to (2.11c), $J_o = (1+\epsilon_o)(1-\nu_a\epsilon_o)^2$. In the following development we shall asume that the strains are small in the sense that $\epsilon_o \ll 1$, and we shall drop all terms which are of order ϵ_o^2 and higher. Thus, using (2.9b), the strain tensor components in the initial configuration are

$$\begin{bmatrix} E_{11}^o & E_{12}^o & E_{13}^o \\ E_{21}^o & E_{22}^o & E_{23}^o \\ E_{31}^o & E_{32}^o & E_{33}^o \end{bmatrix} = \begin{bmatrix} \epsilon_o & 0 & 0 \\ 0 & -\nu_a\epsilon_o & 0 \\ 0 & 0 & -\nu_a\epsilon_o \end{bmatrix}$$

The elastic constants A_{ijkl}^o are obtained with the aid of (4.6) and (4.8), if it is assumed that the medium is isotropic in its natural state. Upon linearization with respect to ϵ_o, they are given by

$$A_{1111}^o = (\lambda_a+2\mu_a) + \epsilon_o[\nu_1^a(1-2\nu_a) + 2\nu_2^a(3-2\nu_a) + 8\nu_3^a]$$

$$A_{2222}^o = (\lambda_a+2\mu_a) + \epsilon_o[\nu_1^a(1-2\nu_a) + 2\nu_2^a(1-4\nu_a) - 8\nu_3^a\nu_a]$$

$$A_{1122}^o = \lambda_a + \epsilon_o[\nu_1^a(1-2\nu_a) + 2\nu_2^a(1-\nu_a)]$$

$$A_{2233}^o = \lambda_a + \epsilon_o[\nu_1^a(1-2\nu_a) - 4\nu_2^a\nu_a]$$

$$A_{1212}^o = \mu_a + \epsilon_o[\nu_2^a(1-2\nu_a) + 2\nu_3^a(1-\nu_a)]$$

[illegible]

Because of the symmetry properties of the tensor component A^{o}_{ijkl}, there are only six independent, non-vanishing elastic constants in the present case. If we now relate the B^{o}_{ijkl} to the A^{o}_{ijkl} and linearize, we readily obtain

B^{o}_{1111} [illegible] (5.12a)

B^{o} [illegible] (5.12b)

B^{o} [illegible] (5.12c)

B^{o} [illegible] (5.12d)

B^{o} [illegible] (5.12e)

B^{o} [illegible] (5.12f)

It can also be shown, to the first order in [illegible], that

λ_{o} [illegible] (5.12g)

μ_{o} [illegible] (5.12h)

[illegible] (5.12i)

We now relate the pre-stress tensor to the initial strain tensor. Using (1.5), (1.6), etc., we obtain, to the first order in [illegible],

[illegible] [7]

$$\Gamma^{o}_{11} = \Lambda^{a}_{1111} E^{o}_{11} + \Lambda^{a}_{11[illegible]} E^{o}_{[illegible]} + \Lambda^{a}_{11[illegible]} E^{o}_{[illegible]}$$

$$= {}_{o}[\Gamma_{a}(1-2[illegible]_{a}) + 2[illegible]_{a}]$$

Conversely, using (2.11b) and linearizing with respect to [illegible], we obtain

$$[illegible]^{o}_{11} = [illegible][1 - {}_{o}(1+2[illegible]_{a})]$$

Upon equating the above expressions, and after linearization, we readily obtain

$$[illegible] = [illegible]_{o}[\Gamma_{a}(1-2\nu_{a})+2[illegible]_{a}] = t_{a}[illegible]_{o}$$

because

$$1-2[illegible]_{a} = \frac{[illegible]_{a}}{[illegible]_{a}+[illegible]_{a}} \; ; \qquad t_{a} = \frac{[illegible]_{a}(3[illegible]_{a}+[illegible])[illegible]_{a}}{[illegible]_{a}+[illegible]_{a}}$$

Upon substitution of (5.1) into (2.18b) and using (2.15), we obtain incremental stress-strain relations

$$\begin{bmatrix} [illegible]_{11} \\ [illegible]_{[illegible]} \\ [illegible]_{[illegible]} \end{bmatrix} = \begin{bmatrix} (B^{o}_{1111}+[illegible]) & (B^{o}_{11[illegible]}-[illegible]) & (B^{o}_{11[illegible]}-[illegible]) \\ B^{o}_{11[illegible]} & B^{o}_{[illegible]} & B^{o}_{[illegible]} \\ B^{o}_{11[illegible]} & B^{o}_{[illegible]} & B^{o}_{[illegible]} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{[illegible]} \\ e_{[illegible]} \end{bmatrix} \quad (5.15a)$$

$$[illegible]_{12} = 2B^{o}_{1212}e_{12} + [illegible]\frac{\partial u_{2}}{\partial x_{1}} \quad (5.15b)$$

$$[illegible]_{23} = 2B^{o}_{2323}e_{23} \quad (5.15c)$$

$$[illegible]_{31} = 2B^{o}_{1313}e_{13} + [illegible]\frac{\partial u_{3}}{\partial x_{1}} \quad (5.15d)$$

Inverting (5.13a), we obtain

$$\begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1^o} & -\dfrac{\eta_1^o}{E_2^o} & -\dfrac{\eta_2^o}{E_2^o} \\ -\dfrac{\eta_3^o}{E_1^o} & \dfrac{1}{E_2^o} & -\dfrac{\eta_2^o}{E_2^o} \\ -\dfrac{\eta_3^o}{E_1^o} & -\dfrac{\eta_2^o}{E_2^o} & \dfrac{1}{E_2^o} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix}$$

where

$$E_1^o = (B_{1111}^o+\sigma) - 2\eta_3^o(B_{1122}^o-\sigma) \tag{5.14a}$$

$$E_2^o = B_{2222}^o - \eta_1^o B_{1122} - \eta_2^o B_{2233} \tag{5.14b}$$

$$\eta_1^o = \frac{(B_{1122}^o-\sigma)(B_{2222}^o-B_{2233}^o)}{[B_{2222}^o(B_{1111}^o+\sigma)-B_{1122}^o(B_{1122}^o-\sigma)]} \tag{5.14c}$$

$$\eta_2^o = \frac{[B_{2233}^o(B_{1111}^o+\sigma)-B_{1122}^o(B_{1122}^o-\sigma)]}{[B_{2222}^o(B_{1111}^o+\sigma)-B_{1122}^o(B_{1122}^o-\sigma)]} \tag{5.14.d}$$

$$\eta_3^o = \frac{B_{1122}^o}{(B_{2222}^o+B_{2233}^o)} \tag{5.14e}$$

The constants (5.14) can be expressed in terms of the material constants of the unstrained, isotropic solid. With the aid of (5.12), and after linearization with respect to ε_o, it can be shown that

$$\begin{aligned} E_1^o = E_a + \frac{\varepsilon_o}{(\lambda_a+\mu_a)^2} \{ & 4\mu_a^3(2+\nu_a)+2\lambda_a\mu_a^2(11+5\nu_a)+3\lambda_a^2\mu_a(5+2\nu_a)+2\nu_a\lambda_a^3 \\ & + \nu_1^a\mu_a^2(1-2\nu_a)+\nu_2^a[3\lambda_a^2(1-2\nu_a)+4\lambda_a\mu_a(2-\nu_a)+2\mu_a^2(3-2\nu_a)] \\ & + 4\nu_3^a[\lambda_a^2(2-\nu_a)+2\mu_a(\mu_a+2\lambda_a)]\} \end{aligned} \tag{5.15a}$$

$$\nu_{13}^{o} = \nu_a \left\{ 1+\sigma_o \left[\frac{2\lambda_a+2\mu_a(1+\nu_a)}{\lambda_a+\mu_a} + \nu_1^a \frac{\mu_a(1-2\nu_a)}{\lambda_a(\lambda_a+\mu_a)} \right.\right.$$

$$\left.\left. + \nu_2^a \frac{\lambda_a(1+4\nu_a)+2\mu_a(1-\nu_a)}{\lambda_a(\lambda_a+\mu_a)} + 4\nu_3^a \frac{\mu_a}{\lambda_a+\mu_a} \right] \right\} \tag{5.15b}$$

etc. If we use the material constants which characterize the elastic behavior of carbon steel as shown in Table 4.1, then $E_1^o=E_a(1-5.24\sigma_o)$ and $\nu_{13}^o=\nu_a(1-4.07\sigma_o)$. We thus conclude that for initial stresses near the yield point, the material constants E_a and ν_a referred to the natural state will change by less than 1%. These small changes are usually neglected in engineering calculations.

In order to obtain the stress-displacement relations for an initially stressed Timoshenko beam, we write

$$\sigma_{11} = E_1^o e_{11} + \nu_1^o \frac{E_1^o}{E_2^o} (\sigma_{22}+\sigma_{33})$$

and upon substitution into (5.5) and ultilization of (2.15), we obtain

$$M = E_1^o I_o \frac{\partial\psi}{\partial x} + \nu_1^o \frac{E_1^o}{E_2^o} \int_{A_o} z(\sigma_{22}+\sigma_{33})dA_o .$$

It is customary to neglect the integral in this expression, so that, approximately,

$$M = E_1^o I_o \frac{\partial\psi}{\partial x} \tag{5.16a}$$

Upon substitution of (5.15d) into (5.5), and utilization of (2.15), we readily obtain

$$V = Q + A_o\sigma_o \frac{\partial w}{\partial x} \tag{5.16b}$$

where

$$Q = A_0 B^0_{1212}(\psi + \frac{\partial w}{\partial x}) \qquad (5.16c)$$

The beam incremental "stress-displacement" relations are given by (5.16). We note that the coefficients in (5.16) are expressed in terms of the material constants of the unstrained medium with the aid of (5.15a), and (5.12e). Moreover, when $\sigma_0 = 0$, equations (5.16) reduce to the corresponding "stress-displacement" relations applicable to a Timoshenko beam without pre-stress [7]. It is customary to multiply the constant B^0_{1212} in (5.16c) by a numerical factor (shear coefficient), and various methods for its determination are discussed in [7]. Accordingly, in the following we shall write (5.16c) in the form

$$Q = A_0 \kappa^2 G_0 (\psi + \frac{\partial w}{\partial x}) \qquad (5.16d)$$

where we set $G_0 = B^0_{1212}$.

[illegible]

We shall now obtain a resolution of the forced motion problem of initially stressed Timoshenko beams in the sense of Section 3. With reference to (5.9) and (5.16), a well posed problem can be stated as follows: We seek a solution $w(x,t)$; $\psi(x,t)$ of the equations

$$\left.\begin{aligned}
&\frac{\partial M}{\partial x} - Q + m = \rho I \frac{\partial^2 \psi}{\partial t^2} \\
&\frac{\partial V}{\partial x} + p = \rho A \frac{\partial^2 w}{\partial t^2} \\
&M = EI \frac{\partial \psi}{\partial x}, \qquad V = Q + \sigma A \frac{\partial w}{\partial x} \\
&Q = A\kappa^2 G(\psi + \frac{\partial w}{\partial x})
\end{aligned}\right\} \qquad (5.17)$$

in $0 < x < \ell$ for $t > 0$. For convenience, we have omitted some of the subscripts previously employed, i.e., $I = I_o$ and $\kappa_o A_o G_o = \kappa A G$. The admissible, non-homogeneous boundary conditions associated with (5.17) are given by

$$
\begin{aligned}
&\text{if } w(o,t) = f_1(t) = w_s(o,t), \text{ then } W_i(o) = 0\\
&\text{if } V(o,t) = f_2(t) = V_s(o,t), \text{ then } V_i(o) = 0\\
&\text{if } \psi(o,t) = f_3(t) = \psi_s(o,t), \text{ then } \Psi_i(o) = 0\\
&\text{if } M(o,t) = f_4(t) = M_s(o,t), \text{ then } M_i(o) = 0\\
&\text{if } w(\ell,t) = g_1(t) = w_s(\ell,t), \text{ then } W_i(\ell) = 0\\
&\text{if } V(\ell,t) = g_2(t) = V_s(\ell,t), \text{ then } V_i(\ell) = 0\\
&\text{if } \psi(\ell,t) = g_3(t) = \psi_s(\ell,t), \text{ then } \Psi_i(\ell) = 0\\
&\text{if } M(\ell,t) = g_4(t) = M_s(\ell,t), \text{ then } M_i(\ell) = 0
\end{aligned}
\tag{5.18}
$$

A sufficient condition for the uniqueness of the solution of the present problem is obtained by the initial conditions

$$
\left.
\begin{aligned}
w(x,o) &= w_o(x), \quad \dot{w}(x,o) = \dot{w}_o(x)\\
\psi(x,o) &= \psi_o(x), \quad \dot{\psi}(x,o) = \dot{\psi}_o(x)
\end{aligned}
\right\}
\tag{5.19}
$$

The solution of the problem characterized by equations (5.17), (5.18), and (5.19) is now written in the form

$$w(x,t) = w_s(x,t) + \sum_{i=1}^{\infty} W_i(x) \cdot q_i(t) \tag{5.20a}$$

$$\psi(x,t) = \psi_s(x,t) + \sum_{i=1}^{\infty} \Psi_i(x) \cdot q_i(t) \tag{5.20b}$$

The quasi-static solution satisfies the equations

$$
\left.
\begin{aligned}
&\frac{\partial V_s}{\partial x} + p = 0 \\
&\frac{\partial M_s}{\partial x} - Q_s + m = 0 \\
&M_s = EI \frac{\partial \psi_s}{\partial x} , \quad V_s = Q_s + \sigma A \frac{\partial w_s}{\partial x} \\
&Q_s = A\kappa^2 G(\psi_s + \frac{\partial w_s}{\partial x})
\end{aligned}
\right\} \qquad (5.21)
$$

The boundary conditions associated with (5.21) are listed in (5.18). The eigenfunctions and eigenvalues satisfy the homogeneous, ordinary differential equations (a prime denotes the derivative with respect to x)

$$
\left.
\begin{aligned}
&\rho I \omega_i^2 \Psi_i + M_i' - Q_i = 0 \\
&\rho A \omega_i^2 W_i + V_i' = 0 \\
&M_i = EI\Psi_i' , \quad V_i = Q_i + \sigma A W_i' \\
&Q_i = A\kappa^2 G(\Psi_i + W_i')
\end{aligned}
\right\} \qquad (5.22)
$$

The eigenfunctions satisfy homogeneous boundary conditions according to (5.18) and they are orthogonal. If, in addition, we normalize them, then

$$
\int_0^l (\rho A W_i W_j + \rho I \Psi_i \Psi_j) dx = \delta_{ij} \qquad (5.23)
$$

where δ_{ij} is the Kronecker delta.

Upon substitution of (5.20) into (5.17) and utilization of (5.22), we obtain

$$\sum_{i=1}^{\infty} (\ddot{q}_i + \omega_i^2 q_i)\rho A W_i = -\rho A \ddot{w}_s \tag{5.24a}$$

$$\sum (\ddot{q}_i + \omega_i^2 q_i)\rho I \Psi_i = -\rho I \ddot{\psi}_s \tag{5.24b}$$

If we set t=0 in (5.20), and substitute the result into (5.19), we obtain

$$\sum_{i=1}^{\infty} W_i(x) \cdot q_i(o) = w_o(x) - w_s(x,o) \tag{5.25a}$$

$$\sum_{i=1}^{\infty} \Psi_i(x) \cdot q_i(o) = \psi_o(x) - \psi_s(x,o) \tag{5.25b}$$

$$\sum_{i=1}^{\infty} W_i(x) \cdot \dot{q}_i(o) = \dot{w}_o(x) - \dot{w}_s(x,o) \tag{5.25c}$$

$$\sum_{i=1}^{\infty} \Psi_i(x) \cdot \dot{q}_i(o) = \dot{\psi}_o(x) - \dot{\psi}_s(x,o) \tag{5.25d}$$

We now multiply (5.24a) and (5.24b) by W_j and Ψ_j, respectively, and add. Similarly, we multiply (5.25a) and (5.25c) by $\rho A W_j$, and (5.25b) and (5.25d) by $\rho I \Psi_j$. Upon application of (5.23), we obtain

$$\ddot{q}_i + \omega_i^2 q_i = \ddot{P}_i(t) \tag{5.26a}$$

$$q_i(o) - P_i(o) = \int_0^{\ell} (\rho A w_o W_i + \rho I \psi_o \Psi_i)dx \tag{5.26b}$$

$$\dot{q}_i(o) - \dot{P}_i(o) = \int_0^{\ell} (\rho A \dot{w}_o W_i + \rho I \dot{\psi}_o \Psi_i)dx \tag{5.26c}$$

where

$$P_i(t) = -\int_0^{\ell} (\rho A w_s W_i + \rho I \psi_s \Psi_i)dx$$

By methods which are analogous to Section 3, it can be shown that

$$\omega_i^2 P_i(t) = (w_s V_i - W_i V_s)\Big|_0^{\ell} + (\psi_s M_i - \Psi_i M_s)\Big|_0^{\ell} - \int_0^{\ell} (\Psi_i m + W_i p)dx \tag{5.27}$$

The solution of [illegible] is given by

$$[illegible]$$

$$[illegible] \qquad (5.[illegible])$$

Thus the solution of the problem posed by eqs. [illegible] (5.[illegible]), and [illegible] is given by [illegible], where W_i, [illegible] $i = 1, 2, \ldots$, is a complete solution set of [illegible] and satisfy [illegible], and the boundary conditions to be satisfied by the eigenfunctions and the quasi-static solution are listed in [illegible]. The [illegible] function [illegible] given by [illegible]. In conjunction with [illegible] (5.[illegible]), and [illegible] complete the formal solution of the general problem.

[illegible]

We now consider the special case of a simply supported uniform beam of length [illegible]. The beam is initially [illegible] and it is at rest and in static equilibrium in its neutral configuration, i.e., $W_0(x) = \dot{W}_0(x) = [illegible]$. At $t = 0$, a constant moment of magnitude M_0 is suddenly applied at $x = L$. Consequently we [illegible] the problem characterized by [illegible] with [illegible] and the boundary conditions [illegible]

$$[illegible]$$

where [illegible] and [illegible] the quasi-static solution is given by the

[illegible]
following.

When

$$\text{[illegible]} \frac{k^2}{(k^2+\text{[illegible]})\,\text{[illegible]}} \;\text{[illegible]}\; 0$$

$$w_s(x,t) = \frac{M_o\,\text{[illegible]}}{EI} \left[\frac{x}{\text{[illegible]}} - \frac{\sinh \text{[illegible]}\,x}{\sinh \text{[illegible]}} \right]$$

$$\psi_s(x,t) = \frac{M_o}{EI} \left[\frac{\cosh \text{[illegible]}\,x}{\text{[illegible]} \sinh \text{[illegible]}} - \text{[illegible]} \right] \qquad (5.30a)$$

When $\text{[illegible]} = \text{[illegible]} = 0$

$$w_s(x,t) = \frac{M_o\,\text{[illegible]}}{6EI} \left(\frac{x}{\text{[illegible]}} - \frac{x}{\text{[illegible]}} \right)$$

$$\psi_s(x,t) = \frac{M_o}{EI} \left(\frac{1}{2} \frac{x^2}{\text{[illegible]}} + \frac{\text{[illegible]}}{k^2} \text{[illegible]} \frac{1}{6} \right) \qquad (5.30b)$$

When $0 < \text{[illegible]} = \text{[illegible]} = \frac{\text{[illegible]}\,k^2}{\text{[illegible]}(k^2+\text{[illegible]})} \text{[illegible]}$

$$w_s(x,t) = \frac{M_o\,\text{[illegible]}}{EI} \left(\frac{x}{\text{[illegible]}} - \frac{\sin \text{[illegible]}\,x}{\sin \text{[illegible]}} \right)$$

$$\psi_s(x,t) = -\frac{M_o}{EI} \left[\frac{\cos \text{[illegible]}\,x}{\text{[illegible]} \sin \text{[illegible]}} + \text{[illegible]} \right] \qquad (5.30c)$$

where $\text{[illegible]} = \frac{I}{A\text{[illegible]}}$, $\text{[illegible]} = \frac{\text{[illegible]}}{E}$, and $k = \frac{\text{[illegible]}G}{E}$

In view of (5.29) and (5.22), the eigenfunctions and associated eigenvalues which satisfy (5.25) and (5.22) are given by

$$W_i(x) = \text{[illegible]}B_i \sin \frac{i\text{[illegible]}x}{\text{[illegible]}} , \quad \text{[illegible]}_i = C_i \cos \frac{i\text{[illegible]}x}{\text{[illegible]}} \qquad (5.31a)$$

where

$$B_i = \sqrt{\frac{2}{\lambda[illegible]}} \; \frac{(i\,k\,[illegible])}{\sqrt{k^4 i^{[illegible]} + [illegible][\,[illegible]_i - i^{[illegible]}(k^2+[illegible])]^2}}$$

$$C_i = \sqrt{\frac{2}{\lambda[illegible]}} \; \frac{[\,[illegible]_i - i^{[illegible]}(k^2+[illegible])]}{\sqrt{k^4 i^{[illegible]} + [illegible][\,[illegible]_i - i^{[illegible]}(k^2+[illegible])]^2}} \qquad (5.31b)$$

$$2[illegible][\,[illegible]_i^{(1)}]^2 = k^2 + ([illegible]i[illegible])^2(1+[illegible]+k^2) + R$$

$$2[illegible][\,[illegible]_i^{(2)}]^2 = k^2 + ([illegible]i[illegible])^2(1+[illegible]+k^2) - R$$

$$R = [k^4 + 2k^2([illegible]i[illegible])^2(1+k^2-[illegible]) + ([illegible]i[illegible])^4(1-[illegible]-k^2[illegible])]^{[illegible]} \qquad (5.31c)$$

$$[illegible]_i = \frac{[illegible]_i}{P}$$

If we set $[\,[illegible]_i^{(r)}]^2 = 0$, r=1,2, in (5.31c), we obtain

$$[illegible]_i = - \frac{([illegible]i[illegible])^2 k^2}{[([illegible]i[illegible])^2 + k^2]}$$

and this is a (dimensionless) formula for the ith static buckling load. When $([illegible]i[illegible])^2 [illegible] k^2$, we have $[illegible]_i$ $[illegible]$ $(i[illegible])$, which is the well known Euler buckling load for a sufficiently slender column. At this point it is appropriate to point out that the subsequent analysis of the forced motion problem is restricted to the case $[illegible]$.

We shall now proceed to synthesize the complete forced motion solution. With the aid of (5.27) and (5.29) we obtain

$$[illegible]_i P_i(t) = - [illegible]_i [illegible] M_s [illegible] = [illegible]_i [illegible] M_o H(t)$$
$$= (-1)^{i+1} C_i M_o H(t)$$

i.e., $\Delta P_i(t)=0$ for $t<0$

$$\Delta P_i(t) = (-1)^{i+1} C_i M_o \quad \text{for } t>0$$

Also, in view of (5.26b), (5.26c), and the given (zero) initial condition, we have

$$q_i(0) - P_i(0) = 0$$

$$\dot{q}_i(0) - \dot{P}_i(0) = 0$$

so that, with reference to (5.28), we have

$$q_i(t) = P_i(t) - \omega_i \int_0^t P_i(\tau)\sin\omega_i(t-\tau)d\tau$$

$$= \frac{1}{\omega_i}(-1)^{i+1} C_i M_o \cos\omega_i t \qquad (5.32)$$

Upon substitution of (5.32) and (5.31a) into (5.20) we obtain the complete forced motion solution

$$w = w^{(s)} + \frac{[illegible]M_o}{E} \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} B_i^{(r)} C_i^{(r)} \frac{(-1)^{i+1}}{[\omega_i^{(r)}]^{2}} \sin i[illegible]\lambda \cos \omega_i^{(r)} T \qquad (5.33a)$$

$$\psi = \psi^{(s)} + \frac{[illegible]M_o}{E} \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} [C_i^{(r)}]^{2} \frac{(-1)^{i+1}}{[\omega_i^{(r)}]^{2}} \cos \frac{i\pi\lambda}{[illegible]} \cos \omega_i^{(r)} T \qquad (5.33b)$$

where $T = \frac{t}{[illegible]}\sqrt{\frac{E}{\rho}}$, the constants $B_i^{(r)}$ and $C_i^{(r)}$ are defined by (5.31b), and (w_s, ψ_s) are given by (5.30).

ACKNOWLEDGEMENT

This research was supported by The United States Air Force Office of Scientific Research under Grant No. AF-AFOSR-76-2915.

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